Course 424

Group Representations III

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EELT 3 Tuesday, 11 May 1999 14:00–16:00

Answer as many questions as you can; all carry the same number of marks.
In this exam, 'Lie algebra' means Lie algebra over $\mathbb{R}$, *and 'representation' means* finite-dimensional representation over $\mathbb{C}$.

1. Define the *exponential* e^X of a square matrix X.

   Determine e^X in each of the following cases:

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$
$$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}.$$

   Which of these 6 matrices X are themselves expressible in the form $X = e^Y$, where Y is a real matrix? (Justify your answers in all cases.)

   Answer: *The exponential of a square matrix X is defined by*

$$e^X = I + X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \cdots .$$

   This series converges for all $X \in \mathbf{Mat}(n, k)$ by comparison with the series for $e^{\|X\|}$, since $\|X^n\| \le \|X\|^n$.

(a) If

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

then

$$X^2 = 0$$

and so

$$e^X = I + X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

(b) If

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

then

$$X^2 = I,$$

and so

$$e^X = \begin{pmatrix} \cosh 1 & \sinh 1 \\ \sinh 1 & \cosh 1 \end{pmatrix}$$

(c) If

$$X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

then

$$X^2 = -I,$$

and so

$$e^X = \begin{pmatrix} \cos 1 & -\sin 1 \\ \sin 1 & \cos 1 \end{pmatrix}.$$

(d) If

$$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

then

$$X = \begin{pmatrix} e & 0 \\ 0 & e^{-1} \end{pmatrix}.$$

(e) If

$$X = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = I + Y,$$

where

$$Y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

then, since I, Y commute,

$$e^X = e^I e^Y = \begin{pmatrix} e\cos 1 & -e\sin 1 \\ e\sin 1 & e\cos 1 \end{pmatrix}.$$

(f) *If*

$$X = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} = -I + Z,$$

where

$$Z = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

then, since $-I, Z$ commute,

$$e^X = e^{-I} e^Z = \begin{pmatrix} e^{-1} & -e^{-1} \\ 0 & e^{-1} \end{pmatrix}.$$

(a) *e^Y is non-singular for all Y, since $e^Y e^{-Y} = I$. Since X is singular in this case, $X \neq e^Y$.*

(b) *X has eigenvalues ± 1. Suppose $X = e^Y$; and suppose Y has eigenvalues λ, μ. Then X has eigenvalues e^λ, e^μ. There are two possibilities. Either λ, μ are complex conjugates, in which case the same is true of e^λ, e^μ; or else λ, μ are both real, in which case $e^\lambda, e^\mu > 0$. In neither case can we get ± 1. Hence $X \neq e^Y$.*

(c) *By the isomorphism between the complex numbers $z = x + iy$ and the matrices*

$$\mathbb{C}(z) = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$$

we see that

$$X = \mathbb{C}(i).$$

Since

$$i = e^{\pi i/2},$$

while

$$\mathbb{C}(e^z) = e^{\mathbb{C}(z)},$$

it follows that $X = e^Y$ with

$$Y = \begin{pmatrix} 0 & -\pi/2 \\ \pi/2 & 0 \end{pmatrix}.$$

(d) *X has eigenvalues ± 1. Thus by the argument in case (b) above, $X \neq e^Y$.*

(e) *We have*

$$X = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \mathbb{C}(1+i).$$

But

$$1 + i = \sqrt{2}e^{\pi i/4} = e^{\log 2/2 + \pi i/4}.$$

Thus $X = e^Y$ *with*

$$Y = X = \begin{pmatrix} \log 2/2 & -\pi/4 \\ \pi/4 & \log 2/2 \end{pmatrix}.$$

(f) X *has eigenvalues* $-1, -1$. *Thus if* $X = e^Y$ *(with* Y *real) then* Y *must have eigenvalues* $\pm(2n+1)\pi i$ *for some integer* n. *In particular,* Y *has distinct eigenvalues, and so is semisimple (diagonalisable over* $\mathbb{C}$*).*

But in that case $X = e^Y$ *would also be semisimple. That is impossible, since a diagonalisable matrix with eigenvalues* $-1, -1$ *is necessarily* $-I$. *Hence* $X \neq e^Y$.

2. Define a *linear group*, and a *Lie algebra*; and define the Lie algebra $\mathscr{L}G$ of a linear group G, showing that it is indeed a Lie algebra.

   Define the *dimension* of a linear group; and determine the dimensions of each of the following groups:

$$\mathbf{O}(n), \mathbf{SO}(n), \mathbf{U}(n), \mathbf{SU}(n), \mathbf{GL}(n,\mathbb{R}), \mathbf{SL}(n,\mathbb{R}), \mathbf{GL}(n,\mathbb{C}), \mathbf{SL}(n,\mathbb{C})?$$

Answer: *A linear group is a closed subgroup* $G \subset \mathbf{GL}(n,\mathbb{R})$ *for some* n.

A Lie algebra is defined by giving

(a) *a vector space* L;

(b) *a binary operation on* L, *ie a map*

$$L \times L \to L : \ (X, Y) \mapsto [X, Y]$$

satisfying the conditions

(a) *The product* $[X, Y]$ *is bilinear in* X, Y;

(b) *The product is skew-symmetric:*

$$[Y, X] = -[X, Y];$$

(c) Jacobi's identity is satisfied:

$$[[X,Y],Z]+[[Y,Z],X]+[[Z,X],Y]=0$$

for all $X,Y,Z \in L$.

Suppose $G \subset \mathbf{GL}(n,\mathbb{R})$ *is a linear group. Then its Lie algebra* $L = \mathcal{L}G$ *is defined to be*

$$L = \{X \in \mathbf{Mat}(n,\mathbb{R}) : e^{tX} \in G \forall t \in \mathbb{R}\}.$$

It follows at once from this definition that

$$X \in L,\ \lambda \in R \implies \lambda X \in L.$$

Thus to see that L *is a vector subspace of* $\mathbf{Mat}(n,\mathbb{R})$ *we must show that*

$$X,Y \in L \implies X+Y \in L.$$

Now

$$\left(e^{X/n}e^{Y/n}\right)^n \mapsto e^{X+Y}$$

as $n \mapsto \infty$. *(This can be seen by taking the logarithms of each side.) It follows that*

$$X,Y \in L \implies e^{X+Y} \in G.$$

On replacing X,Y *by* tX,tY *we see that*

$$\begin{aligned} X,Y \in L &\implies e^{t(X+Y)} \in G \\ &\implies X+Y \in L. \end{aligned}$$

Similarly

$$\left(e^{X/n}e^{Y/n}e^{-X/n}e^{-Y/n}\right)^{n^2} \mapsto e^{[X,Y]},$$

as may be seen again on taking logarithms. It follows that

$$X,Y \in L \implies e^{[X,Y]} \in G.$$

Taking tX *in place of* X, *this implies that*

$$\begin{aligned} X,Y \in L &\implies e^{t[X,Y]} \in G \\ &\implies [X,Y] \in L. \end{aligned}$$

Thus L *is a Lie algebra.*

The dimension of a linear group G *is the dimension of the real vector space* $\mathscr{L}G$:

$$\dim G = \dim_{\mathbb{R}} \mathscr{L}G.$$

(a) *We have*

$$\mathbf{o}(n) = \{X \in \mathbf{Mat}(n, \mathbb{R}) : X' + X = 0\}$$

A skew symmetric matrix X is determined by giving the entries above the diagonal. This determines the entries below the diagonal; while those on the diagonal are 0. Thus

$$\dim O(n) = \dim o(n) = \frac{n(n-1)}{2}.$$

(b) *We have*

$$\mathbf{so}(n) = \{X \in \mathbf{Mat}(n, \mathbb{R}) : X' + X = 0, \operatorname{tr} X = 0\} = \mathbf{o}(n),$$

since $X' + X = 0 \implies \operatorname{tr} X = 0$. Thus

$$\dim SO(n) = \dim O(n) = \frac{n(n-1)}{2}.$$

(c) *We have*

$$\mathbf{u}(n) = \{X \in \mathbf{Mat}(n, \mathbb{C}) : X^* + X = 0\}$$

Again, the elements above the diagonal determine those below the diagonal; while those on the diagonal are purely imaginary. Thus

$$\begin{aligned}\dim \mathbf{U}(n) &= 2\frac{n(n-1)}{2} + n \\ &= n^2.\end{aligned}$$

(d) *We have*

$$\mathbf{su}(n) = \{X \in \mathbf{Mat}(n, \mathbb{C}) : X^* + X = 0,\ \operatorname{tr} X = 0\}$$

This gives one linear condition on the (purely imaginary) diagonal elements. Thus

$$\dim \mathbf{SU}(n) = \dim \mathbf{U}(n) - 1 = n^2 - 1.$$

(e) *We have*

$$\mathbf{gl}(n, \mathbb{R}) = \mathbf{Mat}(n, \mathbb{R}).$$

Thus

$$\dim \mathbf{GL}(n, \mathbb{R}) = n^2.$$

(f) *We have*

$$\mathbf{sl}(n,\mathbb{R}) = \{X \in \mathbf{Mat}(n,\mathbb{R}) : \operatorname{tr} X = 0\}.$$

This imposes one linear condition on X. Thus

$$\dim \mathbf{SL}(n,\mathbb{R}) = \dim \mathbf{GL}(n,\mathbb{R}) - 1 = n^2 - 1.$$

(g) *We have*

$$\mathbf{gl}(n,\mathbb{C}) = \mathbf{Mat}(n,\mathbb{C}).$$

Each of the n^2 complex entries takes 2 real values. Thus

$$\dim \mathbf{GL}(n,\mathbb{C}) = 2n^2.$$

(h) *We have*

$$\mathbf{sl}(n,\mathbb{C}) = \{X \in \mathbf{Mat}(n,\mathbb{C}) : \operatorname{tr} X = 0\}.$$

This imposes one complex linear condition on X, or 2 real linear conditions. Thus

$$\dim \mathbf{SL}(n,\mathbb{C}) = \dim \mathbf{GL}(n,\mathbb{C}) - 1 = 2n^2 - 2.$$

3. Determine the Lie algebras of $\mathbf{SU}(2)$ and $\mathbf{SO}(3)$, and show that they are isomomorphic.

   Show that the 2 groups themselves are *not* isomorphic.

   Answer: *We have*

$$\begin{aligned}\mathbf{u}(2) &= \{X \in \mathbf{Mat}(2,\mathbb{C}) : \ e^{tX} \in \mathbf{U}(2) \forall t \in \mathbb{R}\}\\ &= \{X : (e^{tX})^* e^{tX} = I \forall t\}\\ &= \{X : e^{tX^*} = e^{-tX} = I \forall t\}\\ &= \{X : X^* = -X\}\\ &= \{X : X^* + X = 0\}.\end{aligned}$$

Also

$$\begin{aligned}\mathbf{sl}(2,\mathbb{C}) &= \{X \in \mathbf{Mat}(2,\mathbb{C}) : \ e^{tX} \in \mathbf{SL}(2,\mathbb{C}) \forall t \in \mathbb{R}\}\\ &= \{X : \det e^{tX} = 1 \forall t\}\\ &= \{X : e^{t \operatorname{tr} X} = 1 \forall t\}\\ &= \{X : \operatorname{tr} X = 0\}.\end{aligned}$$

Since

$$\mathbf{SU}(2) = \mathbf{U}(2) \cap \mathbf{SL}(2, \mathbb{C})$$

it follows that

$$\begin{aligned} \mathbf{su}(2) &= \mathbf{u}(2) \cap \mathbf{sl}(2, \mathbb{C}) \\ &= \{X : X^* + X = 0,\ \operatorname{tr} X = 0\}. \end{aligned}$$

The 3 matrices

$$e = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\ f = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ g = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

form a basis for the vector space $\mathbf{su}(2)$*.*

We have

$$\begin{aligned} [e, f] &= ef - fe = -2g, \\ [e, g] &= eg - ge = 2f, \\ [f, g] &= fg - gf = -2e \end{aligned}$$

Thus

$$\mathbf{su}(2) = \langle e, f, g : [e, f] = -2g, [e, g] = 2f, [f, g] = -2e \rangle.$$

We have

$$\begin{aligned} \mathbf{o}(3) &= \{X \in \mathbf{Mat}(3, \mathbb{R}) :\ e^{tX} \in \mathbf{O}(3) \forall t \in \mathbb{R}\} \\ &= \{X : (e^{tX})' e^{tX} = I \forall t\} \\ &= \{X : e^{tX'} = e^{-tX} = I \forall t\} \\ &= \{X : X' = -X\} \\ &= \{X : X' + X = 0\}. \end{aligned}$$

Also

$$\begin{aligned} \mathbf{sl}(3, \mathbb{R}) &= \{X \in \mathbf{Mat}(3, \mathbb{R}) :\ e^{tX} \in \mathbf{SL}(3, \mathbb{R}) \forall t \in \mathbb{R}\} \\ &= \{X : \det e^{tX} = 1 \forall t\} \\ &= \{X : e^{t \operatorname{tr} X} = 1 \forall t\} \\ &= \{X : \operatorname{tr} X = 0\}. \end{aligned}$$

Since

$$\mathbf{SO}(3) = \mathbf{O}(3) \cap \mathbf{SL}(3, \mathbb{R})$$

it follows that

$$\begin{aligned}\mathbf{so}(3) &= \mathbf{o}(3) \cap \mathbf{sl}(3, \mathbb{R}) \\ &= \{X : X' + X = 0,\ \operatorname{tr} X = 0\} \\ &= \{X : X' + X = 0\}\end{aligned}$$

since a skew-symmetric matrix necessarily has trace 0.

The 3 matrices

$$U = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},\ V = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},\ W = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

form a basis for the vector space $\mathbf{so}(3)$*.*

We have

$$[V, W] = U;$$

and so by cyclic permutation of indices (or coordinates)

$$[W, U] = V,\ [U, V] = W.$$

Thus

$$\mathbf{so}(3) = \langle U, V, W : [U, V] = W, [U, W] = -V, [V, W] = U \rangle.$$

Finally, $su(2)$ *and* $so(3)$ *are isomorphic under the correspondence*

$$e \leftrightarrow -2U,\ f \leftrightarrow -2V,\ g \leftrightarrow -2W.$$

However, the groups $\mathbf{SU}(2), \mathbf{SO}(3)$ *are not isomorphic, since*

$$Z\mathbf{SU}(2) = \{\pm I\} \text{ while } Z\mathbf{SO}(3) = \{I\}.$$

4. Define a *representation* of a Lie algebra; and show how each representation α of a linear group G gives rise to a representation $\mathscr{L}\alpha$ of $\mathscr{L}G$.

   Determine the Lie algebra of $\mathbf{SL}(2, \mathbb{R})$; and show that this Lie algebra $\mathbf{sl}(2, \mathbb{R})$ has just 1 simple representation of each dimension $1, 2, 3, \ldots$.

   Answer: *Suppose* L *is a real Lie algebra. A representation of* L *in the complex vector space* V *is defined by giving a map*

$$L \times V \to V :\ (X, v) \mapsto Xv$$

which is bilinear over $\mathbb{R}$ *and which satisfies the condition*

$$[X,Y]v = X(Yv) - Y(Xv)$$

for all $X, Y \in L, v \in V$*.*

A representation of L *in* V *is thus the same as a representation of the complexification* $L_{\mathbb{C}}$ *of* L *in* V*.*

Suppose α *is a representation of the linear group* G*, ie a homomorphism*

$$\alpha : G \to \mathbf{GL}(n, \mathbb{C}).$$

Under the Lie correspondence this gives rise to a Lie algebra homomorphism

$$A = \mathscr{L}\alpha : L = \mathscr{L}G \to \mathbf{gl}(n, \mathbb{C}).$$

But now L *acts on* $V = C^n$ *by*

$$Xv = A(X)v.$$

This defines a representation of L *in* V *since*

$$\begin{aligned}[X,Y]v &= A([X,Y])v \\ &= [AX, AY]v \\ &= ((AX)(AY) - (AY)(AX))v \\ &= X(Yv) - Y(Xv).\end{aligned}$$

We have

$$\begin{aligned}\mathbf{sl}(2,\mathbb{R}) &= \{X \in \mathbf{Mat}(2,\mathbb{R}) : \ e^{tX} \in \mathbf{SL}(2,\mathbb{R}) \forall t \in \mathbb{R}\} \\ &= \{X : \det e^{tX} = 1 \forall t\} \\ &= \{X : e^{t \operatorname{tr} X} = 1 \forall t\} \\ &= \{X : \operatorname{tr} X = 0\}.\end{aligned}$$

The 3 matrices

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

form a basis for the vector space $\mathbf{sl}(2,\mathbb{R})$*.*

We have

$$[H,E] = 2E, \ [H,F] = -2F, \ [E,F] = H.$$

Thus

$$\mathbf{sl}(2,\mathbb{R}) = \langle H, E, F : [H,E] = 2E, [H,F] = -2F, [V,W] = H\rangle.$$

Now suppose we have a simple representation of $\mathbf{sl}(2,\mathbb{R})$ *on* V*. Suppose* v *is an eigenvector of* H *with eigenvalue* λ*:*

$$Hv = \lambda v.$$

Now

$$[H,E]v = 2Ev,$$

that is,

$$HEv - EHv = 2Ev.$$

In other words, since $Hv = \lambda v$*,*

$$H(Ev) = (\lambda + 2)Ev,$$

ie Ev *is an eigenvector of* H *with eigenvalue* $\lambda + 2$*.*

By the same argument $E^2v, E^3v, \ldots$ *are all eigenvectors of* H *with eigenvalues* $\lambda + 4, \lambda + 6, \ldots,$ *at least until they vanish.*

This must happen at some point, since V *is finite-dimensional; say*

$$E^{r+1}v = 0, E^r v \neq 0.$$

Similarly we find that

$$Fv, F^2v, \ldots$$

are also eigenvectors of H *(until they vanish) with eigenvalues* $\lambda-2, \lambda-4, \ldots$*. Again we must have*

$$F^{s+1}v = 0, F^s v \neq 0$$

for some s*.*

Now let us write e_0 *for* $F^s v$*, so that*

$$Fe_0 = 0;$$

and let us set

$$e^i = E^i e_0.$$

Then the e_i are all eigenvectors of H. Let us set $e_i = 0$ for i outside the range $[0, n-1]$. Suppose e_0 is a μ-eigenvector. Then e_i is a $(\mu + 2i)$-egenvector. Let us suppose that there are n eigenvectors in the sequence, ie

$$e_{n-1} \neq 0, \ Ee_{n-1} = 0.$$

Now we show by induction that

$$Fe_i = \rho_i e_{i-1}$$

for each i. The result holds for $i = 0$ with $\rho_0 = 0$. Suppose it holds for $i = 1, 2, \ldots, m$. Then

$$\begin{aligned} Fe_{m+1} &= FEe_m \\ &= (EF - [E, F])e_m \\ &= \rho_m Ee_{m-1} - He_m \\ &= (\rho_m - \mu - 2m)e_m. \end{aligned}$$

This proves the result, and also shows that

$$\rho_{i+1} = \rho_i - \mu - 2i$$

for each i. It follows that

$$\rho_i = -i\mu - i(i-1).$$

We must have $\rho_n = 0$. Hence

$$\mu = n - 1.$$

We conclude that the subspace

$$\langle e_0, \ldots, e_{n-1} \rangle$$

is stable under $\mathbf{sl}(2, \mathbb{R})$, and so must be the whole of V.

Thus we have shown that there is at most 1 simple representation of each dimension n, and we have determined this explicitly, if it exists. In fact it is a straightforward matter to verify that the above actions of H, E, F on $\langle e_0, \ldots, e_{n-1} \rangle$ do indeed define a representation of $\mathbf{sl}(2, \mathbb{R})$; so that this Lie algebra has exactly 1 simple representation of each dimension.

5. Show that a compact connected abelian linear group of dimension n is necessarily isomorphic to the torus $\mathbb{T}^n$.

Answer: *If G is a abelian linear group then $\mathscr{L}G$ is trivial, ie $[X, Y] = 0$ for all $X, Y \in \mathscr{L}G$. For $e^{tX}, e^{tY} \in G$ commute for all t. If t is sufficiently small we can take logs, and deduce that $tX = \log(e^{tX}), tY = \log(e^{tY})$ commute. Hence X, Y commute.*

The map

$$\Theta : \mathscr{L}G \to G$$

under which

$$X \mapsto e^X$$

is a homomorphism, since

$$X + Y \mapsto e^{X+Y} = e^X e^Y.$$

For any linear group G, there exist open subsets $U \ni 0$ in $\mathscr{L}G$, $V \ni I$ in G such that $X \mapsto e^X$ defines a homeomorphism $U \to V$.

It follows that $\operatorname{im}\Theta \subset \mathscr{L}G$ is an open subgroup of G. Since G is connected, $\operatorname{im}\Theta = G$. Thus

$$G \cong \mathscr{L}G / \ker\Theta.$$

Moreover, $\ker\Theta$ is discrete, since $U \cap \ker\Theta = \{0\}$. Thus

$$G \cong \mathbb{R}^n / K,$$

where K is a discrete subgroup, and $n = \dim G$.

Lemma: *A discrete subgroup $K \subset \mathbb{R}^n$ is necessarily $\cong \mathbb{Z}^d$ for some $d \leq n$, ie we can find a $\mathbb{Z}$-basis $k_1, \dots, k_d$ for K such that*

$$K = \{n_1 k_1 + \cdots + n_d k_d : n_1, \dots, n_d \in \mathbb{Z}\}.$$

Proof: *Let k_1 be one of the closest points to 0 in $K \setminus \{0\}$. Then let k_2 be one of the closest points to the subspace $\langle k_1 \rangle$ in $K \setminus \langle k_1 \rangle$, let k_3 be one of the closest points to the subspace $\langle k_1, k_2 \rangle$ in $K \setminus \langle k_1, k_2 \rangle$, and so on.*

Then $k_1, k_2, \dots$ are linearly independent. So the process must end after $d \leq n$ steps:

$$K = \langle k_1, \dots, k_d \rangle.$$

Now suppose $k \in K$, say

$$k = \lambda_1 k_1 + \cdots + \lambda_d k_d.$$

We show that $\lambda_1, \ldots, \lambda_d \in \mathbb{Z}$. Let

$$\lambda_d = r + \epsilon,$$

where $r \in \mathbb{Z}$ and $|\epsilon| \le 1/2$. Then

$$k - rk_d = \lambda_1 k_1 + \cdots + \lambda_{d-1} k_{d-1} + \epsilon k_d$$

is closer to $\langle k_1, \ldots, k_{d-1} \rangle$ than is k_d. Hence $\epsilon = 0$, ie $\lambda_d \in \mathbb{Z}$.
Applying the same argument to

$$k - \lambda_d k_d = \lambda_1 k_1 + \cdots + \lambda_{d-1} k_{d-1},$$

we deduce that $\lambda_{d-1} \in \mathbb{Z}$; and so successively $\lambda_{d-2}, \ldots, \lambda_1 \in \mathbb{Z}$
Thus $k_1, \ldots, k_d$ is a Z-basis for K. Extend $k_1, \ldots, k_d$ to a basis $k_1, \ldots, k_d, e_1, \ldots, e_{n-d}$ of $\mathbb{R}^n$. Then

$$\begin{aligned} G &\cong \mathbb{R}^n/K \cong \mathbb{R}/\mathbb{Z} \oplus \cdots \oplus \mathbb{R}/\mathbb{Z} \\ &\cong \mathbb{T}^d \oplus \mathbb{R}^{n-d}. \end{aligned}$$

Since G is compact, $n - d = 0$, ie

$$G \cong \mathbb{T}^n.$$