Course 424

Group Representations IIc

Dr Timothy Murphy

Mathematics 1.8 Monday, 19 April 1999 16:00–17:30

Answer as many questions as you can; all carry the same number of marks.
All representations are finite-dimensional over $\mathbb{C}$.

1. What is meant by a *measure* on a compact space X? What is meant by saying that a measure on a compact group G is *invariant*? Sketch the proof that every compact group G carries such a measure.

   Prove that every representation of a compact group is semisimple.

2. Which of the following groups are (a) abelian, (b) compact, (c) connected:

   $\mathbf{SO}(2), \mathbf{O}(2), \mathbf{U}(2), \mathbf{SU}(2), \mathbf{Sp}(2), \mathbf{GL}(2, \mathbb{R}), \mathbf{SL}(2, \mathbb{R}), \mathbf{GL}(2, \mathbb{C}), \mathbf{SL}(2, \mathbb{C}), \mathbb{T}^2$?

   Justify your answer in each case; no marks will be given for unsupported assertions.

3. Prove that every simple representation of a compact abelian group is 1-dimensional and unitary.

   Determine the simple representations of $\mathbf{U}(1)$.

   Determine also the simple representations of $\mathbf{O}(2)$.

4. Determine the conjugacy classes in $\mathbf{SU}(2)$; and prove that this group has just one simple representation of each dimension.

   Find the character of the representation $D(j)$ of dimensions $2j + 1$ (where $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$).

   Determine the representation-ring of $\mathbf{SU}(2)$, ie express each product $D(i)D(j)$ as a sum of simple representations $D(k)$.

5. Show that there exists a surjective homomorphism

   $$\Theta : \mathbf{SU}(2) \to \mathbf{SO}(3)$$

   with finite kernel.

   Hence or otherwise determine all simple representations of $\mathbf{SO}(3)$.

   Determine also all simple representations of $\mathbf{O}(3)$.