Course 424

Group Representations IIa

Dr Timothy Murphy

Mathematics 1.8 Friday, 9 April 1999 16:00–17:30

Answer as many questions as you can; all carry the same number of marks.
All representations are finite-dimensional over $\mathbb{C}$.

1. What is meant by a *measure* on a compact space X? What is meant by saying that a measure on a compact group G is *invariant*? Sketch the proof that every compact group G carries such a measure. To what extent is this measure unique?

2. Determine the conjugacy classes in $\mathbf{SU}(2)$. Prove that $\mathbf{SU}(2)$ has just one simple representation of each dimension $1, 2, 3, \ldots$; and determine the character of this representation.

3. Show that there exists a surjective homomorphism

$$\Theta : \mathbf{SU}(2) \to \mathbf{SO}(3)$$

with finite kernel.

Hence or otherwise determine all simple representations of $\mathbf{SO}(3)$.

4. Explain the division of simple representations of a compact group G into *real*, *essentially complex* and *quaternionic*.

Show that the representations of $\mathbf{SO}(3)$ are all real.

5. Define the exterior product $\lambda^r\alpha$ of a representation α.

Show that the character of $\lambda^2\alpha$ is given by

$$\chi_{\lambda^2\alpha}(g) = \frac{1}{2}\left(\chi_\alpha(g)^2 - \chi_\alpha(g^2)\right).$$

Hence or otherwise show that if $D(j)$ is the representation of $\mathbf{SU}(2)$ of dimension $2j+1$ then

$$\lambda^2 D(j) = D(2j-1) + D(2j-3) + \cdots + \begin{cases} D(1) & \text{if } j \text{ is integral} \\ D(0) & \text{if } j \text{ is half-integral} \end{cases}.$$