Course 424

Group Representations II

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EELT3 Tuesday, 13 April 1999 16:00–17:30

Answer as many questions as you can; all carry the same number of marks.
All representations are finite-dimensional over $\mathbb{C}$*.*

1. What is meant by a *measure* on a compact space X? What is meant by saying that a measure on a compact group G is *invariant*? Sketch the proof that every compact group G carries such a measure. To what extent is this measure unique?

2. Prove that every simple representation of a compact abelian group is 1-dimensional and unitary.

   Determine the simple representations of $\mathbf{SO}(2)$.

   Determine also the simple representations of $\mathbf{O}(2)$.

3. Determine the conjugacy classes in $\mathbf{SU}(2)$; and prove that this group has just one simple representation of each dimension.

   Find the character of the representation $D(j)$ of dimensions $2j+1$ (where $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$).

   Determine the representation-ring of $\mathbf{SU}(2)$, ie express each product $D(i)D(j)$ as a sum of simple representations $D(k)$.

4. Show that there exists a surjective homomorphism

   $$\Theta : \mathbf{SU}(2) \to \mathbf{SO}(3)$$

   with finite kernel.

   Hence or otherwise determine all simple representations of $\mathbf{SO}(3)$.

   Determine also all simple representations of $\mathbf{O}(3)$.

5. Explain the division of simple representations of a finite or compact group G over $\mathbb{C}$ into *real*, *essentially complex* and *quaternionic*. Give an example of each (justifying your answers).

   Show that if α is a simple representation with character χ then the value of

   $$\int_G \chi(g^2)\, dg$$

   determines which of these three types α falls into.

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