Course 424

Group Representations II

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EELT3 Tuesday, 13 April 1999 16:00–17:30

Answer as many questions as you can; all carry the same number of marks.
All representations are finite-dimensional over $\mathbb{C}$.

1. What is meant by a *measure* on a compact space X? What is meant by saying that a measure on a compact group G is *invariant*? Sketch the proof that every compact group G carries such a measure. To what extent is this measure unique?

   Answer: *A* measure μ *on* X *is a continuous linear functional*

   $$\mu : C(X) \to \mathbb{C},$$

   where $C(X) = C(X, \mathbb{R})$ *is the space of real-valued continuous functions on* X *with norm* $\|f\| = \sup |f(x)|$.

   The compact group G *acts on* $C(G)$ *by*

   $$(gf)(x) = f(g^{-1}x).$$

   The measure μ *is said to be invariant under* G *if*

   $$\mu(gf)\mu(f)$$

   for all $g \in G,\ f \in C(G)$.

   By an average F *of* $f \in C(G)$ *we mean a function of the form*

   $$F = \lambda_1 g_1 f + \lambda_2 g_2 f + \cdots + \lambda_r g_r f,$$

   where $0 \le \lambda_i \le 1,\ \sum \lambda_i = 1$ *and* $g_1, g_2, \ldots, g_r \in G$.

   If F *is an average of* f *then*

   (a) $\inf f \le \inf F \le \sup F \le supf$*;*

   (b) If μ *is an invariant measure then* $\mu(F) = \mu(f)$*;*

   (c) An average of F *is an average of* f.

If we set

$$\operatorname{var}(f) = \sup f - \inf f$$

then

$$\operatorname{var}(F) \leq \operatorname{var}(f)$$

for any average F *of* f. *We shall establish a sequence of averages* $F_0 = f, F_1, F_2, \ldots$ *(each an average of its predecessor) such that* $\operatorname{var}(F_i) \to 0$. *It follows that*

$$F_i \to c \in \mathbb{R},$$

ie $F_i(g) \to c$ *for each* $g \in G$.

Suppose $f \in C(G)$. *It is not hard to find an average* F *of* f *with* $\operatorname{var}(F) < \operatorname{var}(f)$. *Let*

$$V = \{g \in G : f(g) < \frac{1}{2}(\sup f + \inf f),$$

ie V *is the set of points where* f *is 'below average'. Since* G *is compact, we can find* $g_1, \ldots, g_r$ *such that*

$$G = g_1 V \cup \cdots \cup g_r V.$$

Consider the average

$$F = \frac{1}{r}\left(g_1 f + \cdots + g_r f\right).$$

Suppose $x \in G$. *Then* $x \in g_i V$ *for some* i, *ie*

$$g_i^{-1} x \in V.$$

Hence

$$(g_i f)(x) < \frac{1}{2}(\sup f + \inf f),$$

and so

$$\begin{aligned} F(x) &< \frac{r-1}{r} \sup f + \frac{1}{2r}(\sup f + \inf f) \\ &= \sup f - \frac{1}{2r} \sup f - \inf f. \end{aligned}$$

Hence $\sup F < supf$ *and so*

$$\operatorname{var}(F) < \operatorname{var}(f).$$

This allows us to construct a sequence of averages $F_0 = f, F_1, F_2, \ldots$ *such that*

$$\operatorname{var}(f) = \operatorname{var}(F)_0 > \operatorname{var}(F)_1 > \operatorname{var}(F)_2 > \cdots.$$

But that is not sufficient to show that $\mathrm{var}(F)_i \to 0$*. For that we must use the fact that any* $f \in C(G)$ *is* uniformly *continuous.*

[I would accept this last remark as sufficient in the exam, and would not insist on the detailed argument that follows.]

In other words, given $\epsilon > 0$ *we can find an open set* $U \ni e$ *such that*

$$x^{-1}y \in U \implies |f(x) - f(y)| < \epsilon.$$

Since

$$(g^{-1}x)^{-1}(g^{-1}y) = x^{-1}y,$$

the same result also holds for the function gf*. Hence the result holds for any average* F *of* f*.*

Let V *be an open neighbourhood of* e *such that*

$$VV \subset U, \quad V^{-1} = V.$$

(If V *satisfies the first condition, then* $V \cap V^{-1}$ *satisfies both conditions.) Then*

$$xV \cup yV \neq \emptyset \implies |f(x) - f(y)| < \epsilon.$$

For if $xv = yv'$ *then*

$$x^{-1}y = vv'^{-1} \in U.$$

Since G *is compact we can find* $g_1, \ldots, g_r$ *such that*

$$G = g_1V \cup \cdots \cup g_rV.$$

Suppose f *attains its minimum* $\inf f$ *at* $x_0 \in g_iV$*; and suppose* $x \in g_jV$*. Then*

$$g_i^{-1}x_0,\ g_j^{-1}x \in V.$$

Hence

$$\left(g_j^{-1}x\right)^{-1}\left(g_i^{-1}x_0\right) = \left(g_ig_j^{-1}x\right)^{-1}x_0 \in U,$$

and so

$$|f(g_ig_j^{-1}x) - f(x_0)| < \epsilon.$$

In particular,

$$(g_jg_i^{-1}f)(x) < \inf f + \epsilon.$$

Let F *be the average*

$$F = \frac{1}{r^2}\sum_{i,j} g_jg_i^{-1}f.$$

Then

$$\sup F < \frac{r^2-1}{r^2}\sup f + \frac{1}{r^2}(\inf f + \epsilon),$$

and so

$$\begin{aligned}\operatorname{var}(F) &< \frac{r^2-1}{r^2}\operatorname{var}(f) + \frac{1}{r^2}\epsilon \\ &< \frac{r^2-1/2}{r^2}\operatorname{var}(f),\end{aligned}$$

if $\epsilon < \operatorname{var}(f)/2$.

Moreover this result also holds for any average of f *in place of* f. *It follows that a succession of averages of this kind*

$$F_0 = f, F_1, \ldots, F_s$$

will bring us to

$$\operatorname{var}(F)_s < \frac{1}{2}\operatorname{var}(f).$$

Now repeating the same argument with F_s, *and so on, we will obtain a sequence of successive averages* $F_0 = f, F_1, \ldots$ *with*

$$\operatorname{var}(F)_i \downarrow 0.$$

It follows that

$$F_i \to c$$

(the constant function with value c*).*

It remains to show that this limit value c *is unique. For this we introduce right averages*

$$H(x) = \sum_j \mu_j f(xh_j)$$

where $0 \le \mu_j \le 1$, $\sum \mu_j = 1$. *(Note that a right average of* f *is in effect a left average of* $\tilde{f}$, *where* $\tilde{f}(x) = f(x^{-1})$. *In particular the results we have established for left averages will hold equally well for right averages.)*

Given a left average and a right average of f, *say*

$$F(x) = \sum \lambda_i f(g_i^{-1}x), \quad H(x) = \sum \mu_j f(xh_j),$$

we can form the joint average

$$J(x) = \sum_{i,j} \lambda_i \mu_j f(g_i^{-1} x h_j).$$

It is easy to see that

$$\begin{aligned}\inf F \le \inf J \le \sup J \le \sup H,\\ \sup F \ge \sup J \ge \inf J \ge \inf H.\end{aligned}$$

But if now $H_0 = f, H_1, \ldots$ is a succession of right averages with $H_i \to d$ then it follows that

$$c = d.$$

In particular, any two convergent sequences of successive left averages must tend to the same limit. We can therefore set

$$\mu(f) = c.$$

Thus $\mu(f)$ is well-defined; and it is invariant since f and gf have the same set of averages. Finally, if $f = 1$ then $\mathrm{var}(f) = 0$, and $f, f, f, \ldots$ converges to 1, so that

$$\mu(1) = 1.$$

The invariant measure on G is unique up to a scalar multiple. In other words, it is unique if we normalise the measure by specifying that

$$\mu(1) = 1$$

(where 1 on the left denotes the constant function 1).

2. Prove that every simple representation of a compact abelian group is 1-dimensional and unitary.

   Determine the simple representations of $\mathbf{SO}(2)$.

   Determine also the simple representations of $\mathbf{O}(2)$.

   Answer: *Suppose α is a simple representation of the compact abelian group G in V.*

   Suppose $g \in G$. Let λ be an eigenvalue of g, and let $E = E_\lambda$ be the corresponding eigenspace. We claim that E is stable under G. For suppose $h \in G$. Then

$$e \in E \implies g(he) = h(ge) = \lambda he \implies he \in E.$$

   Since α is simple, it follows that $E = V$, ie $gv = \lambda v$ for all v, or $g = \lambda I$.

   Since this is true for all $g \in G$, it follows that every subspace of V is stable under G. Since α is simple, this implies that $\dim V = 1$, ie α is of degree 1.

   Thus a simple representation of G is a homomorphism $\alpha : G \to \mathbb{C}^*$. We must show that

$$|\alpha(g)| = 1$$

   for all $g \in G$.

   If $|\alpha(g)| > 1$ then

$$|\alpha(g^n)| = (|g|)^n \to \infty.$$

This is a contradiction, since $\operatorname{im}\alpha \subset \mathbb{C}^*$ *is compact and so bounded. On the other hand, if* $|\alpha(g)| < 1$ *then* $|\alpha(g^{-1})| > 1$. *Hence* $|\alpha(g)| = 1$ *for all* g, *ie* α *is unitary.*

We can identify $\mathbf{SO}(2)$ *with*

$$\mathbf{U}(1) = \{z \in \mathbb{C} : |z| = 1\}.$$

From above, a representation of $\mathbf{U}(1)$ *is a homomorphism*

$$\alpha : \mathbf{U}(1) \to \mathbf{U}(1).$$

For each $n \in \mathbb{Z}$ *the map*

$$E(n) : z \to z^n$$

defines such a homomorphism. We claim that every representation of $\mathbf{U}(1)$ *is of this form.*

3. Determine the conjugacy classes in $\mathbf{SU}(2)$; and prove that this group has just one simple representation of each dimension.

   Find the character of the representation $D(j)$ of dimensions $2j+1$ (where $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$).

   Determine the representation-ring of $\mathbf{SU}(2)$, ie express each product $D(i)D(j)$ as a sum of simple representations $D(k)$.

   Answer: *We know that*

   (a) if $U \in \mathbf{SU}(2)$ *then* U *has eigenvalues*

   $$e^{\pm i\theta} \; (\theta \in \mathbb{R}).$$

   (b) if $X, Y \in \mathbf{GL}(n, k)$ *then*

   $$X \sim Y \implies X, Y \text{ have the same eigenvalues.}$$

   A fortiori, if $U \sim V \in \mathbf{SU}(2)$ *then* U, V *have the same eigenvalues.*

   We shall show that the converse of the last result is also true, that is: $U \sim V$ *in* $\mathbf{SU}(2)$ *if and only if* U, V *have the same eigenvalues* $e^{\pm i\theta}$, *This is equivalent to proving that*

   $$U \sim U(\theta) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix},$$

   ie we can find $V \in \mathbf{SU}(2)$ *such that*

   $$V^{-1}UV = U(\theta).$$

To see this, let v be an $e^{i\theta}$-eigenvalue of U. Normalise v, so that $v^*v = 1$; and let w be a unit vector orthogonal to v, ie $w^*w = 1,\ v^*w = 0$. Then the matrix

$$V = (vw) \in \mathbf{Mat}(2, \mathbb{C})$$

is unitary; and

$$V^{-1}UV = \begin{pmatrix} e^{i\theta} & x \\ 0 & e^{-i\theta} \end{pmatrix}$$

But in a unitary matrix, the squares of the absolute values of each row and column sum to 1. It follows that

$$|e^{i\theta}|^2 + |x|^2 = 1 \implies x = 0,$$

ie

$$V^{-1}UV = U(\theta).$$

We only know that $V \in \mathbf{U}(2)$, not that $V \in \mathbf{SU}(2)$. However

$$V \in \mathbf{U}(2) \implies |\det V| = 1 \implies \det V = e^{i\phi}.$$

Thus

$$V' = e^{-i\phi/2}V \in \mathbf{SU}(2)$$

and still

$$(V')^{-1}UV = U(\theta).$$

To summarise: Since $U(-\theta) \sim U(\theta)$ (by interchange of coordinates), we have show that if

$$C(\theta) = \{U \in \mathbf{SU}(2) : U \text{ has eigenvalues } e^{\pm i\theta}\}$$

then the conjugacy classes in $\mathbf{SU}(2)$ are

$$C(\theta) \quad (0 \leq \theta \leq \pi).$$

Now suppose $m \in \mathbb{N}$, Let $V(m)$ denote the space of homogeneous polynomials $P(z, w)$ in z, w. Thus $V(m)$ is a vector space over $\mathbb{C}$ of dimension $m + 1$, with basis $z^m, z^{m-1}w, \ldots, w^m$.

Suppose $U \in \mathbf{SU}(2)$. Then U acts on z, w by

$$\begin{pmatrix} z \\ w \end{pmatrix} \mapsto \begin{pmatrix} z' \\ w' \end{pmatrix} = U \begin{pmatrix} z \\ w \end{pmatrix}.$$

This action in turn defines an action of $\mathbf{SU}(2)$ on $V(m)$:

$$P(z, w) \mapsto P(z', w').$$

We claim that the corresponding representation of $\mathbf{SU}(2)$ *— which we denote by* $D_{m/2}$ *— is simple, and that these are the only simple (finite-dimensional) representations of* $\mathbf{SU}(2)$ *over* $\mathbb{C}$.

To prove this, let

$$\mathbf{U}(1) \subset \mathbf{SU}(2)$$

be the subgroup formed by the diagonal matrices $U(\theta)$. *The action of* $\mathbf{SU}(2)$ *on* z, w *restricts to the action*

$$(z, w) \mapsto (e^{i\theta} z, e^{-i\theta} w)$$

of $\mathbf{U}(1)$. *Thus in the action of* $\mathbf{U}(1)$ *on* $V(m)$,

$$z^{m-r} w^r \mapsto e^{(m-2r)i\theta} z^{m-r} w^r,$$

It follows that the restriction of $D_{m/1}$ *to* $U(1)$ *is the representation*

$$D_{m/2}|\mathbf{U}(1) = E(m) + E(m-2) + \cdots + E(-m)$$

where $E(m)$ *is the representation*

$$e^{i\theta} \mapsto e^{mi\theta}$$

of $\mathbf{U}(1)$.

In particular, the character of $D_{m/2}$ *is given by*

$$\chi_{m/2}(U) = e^{mi\theta} + e^{(m-2} i\theta + \cdots + e^{-mi\theta}$$

if U *has eigenvalues* $e^{\pm i\theta}$.

Now suppose $D_{m/2}$ *is* not *simple, say*

$$D_{m/2} = \alpha + \beta.$$

(We know that $D_{m/2}$ *is semisimple, since* $\mathbf{SU}(2)$ *is compact.) Let a corresponding split of the representation space be*

$$V(m) = W_1 \oplus W_2.$$

Since the simple parts of $D_{m/2}|\mathbf{U}(1)$ *are distinct, the expression of* $V(m)$ *as a direct sum of* $\mathbf{U}(1)$*-spaces,*

$$V(m) = \langle z^m \rangle \oplus \langle z^{m-1} w \rangle \oplus \cdots \oplus \langle w^m \rangle$$

is unique. It follows that W_1 *must be the direct sum of some of these spaces, and* W_2 *the direct sum of the others. In particular* $z^m \in W_1$ *or* $z^n \in W_2$, *say* $z^m \in W_1$. *Let*

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \in \mathbf{SU}(2).$$

Then

$$\begin{pmatrix} z \\ w \end{pmatrix} \mapsto \frac{1}{\sqrt{2}} \begin{pmatrix} z + w \\ -z + w \end{pmatrix}$$

under U. Hence

$$z^m \mapsto 2^{-m/2}(z+w)^m.$$

Since this contains non-zero components in each subspace $\langle z^{m-r} w^r \rangle$, it follows that

$$W_1 = V(m),$$

ie the representation $D_{m/2}$ of $\mathbf{SU}(2)$ in $V(m)$ is simple.

To see that every simple (finite-dimensional) representation of $\mathbf{SU}(2)$ is of this form, suppose α is such a representation. Consider its restriction to $\mathbf{U}(1)$. Suppose

$$\alpha|\mathbf{U}(1) = e_r E(r) + e_{r-1} E(r-1) + \cdots + e_{-r} E(-r) \quad (e_r, e_{r-1}, \ldots, e_{-r} \in \mathbb{N}).$$

Then α has character

$$\chi(U) = \chi(\theta) = e_r e^{ri\theta} + e_{r-1} e^{(r-1)i\theta} + \cdots + e_{-r} e^{-ri\theta}$$

if U has eigenvalues $e^{\pm i\theta}$.

Since $U(-\theta) \sim U(\theta)$ it follows that

$$\chi(-\theta) = \chi(\theta),$$

and so

$$e_{-i} = e_i,$$

ie

$$\chi(\theta) = e_r(e^{ri\theta} + e^{-ri\theta}) + e_{r-1}(e^{(r-1)i\theta} + e^{-(r-1)i\theta}) + \cdots .$$

It is easy to see that this is expressible as a sum of the $\chi_j(\theta)$ with integer (possibly negative) coefficients:

$$\chi(\theta) = a_0 \chi_0(\theta) + a_{1/2} \chi_{1/2}(\theta) + \cdots + a_s \chi_s(\theta) \quad (a_0, a_{1/2}, \ldots, a_s \in \mathbb{Z}).$$

Using the intertwining number,

$$I(\alpha, \alpha) = a_0^2 + a_{1/2}^2 + \cdots + a_s^2$$

(since $I(D_j, D_k) = 0$). Since α is simple,

$$I(\alpha, \alpha) = 1.$$

It follows that one of the coefficients a_j is ± 1 and the rest are 0, ie

$$\chi(\theta) = \pm\chi_j(\theta)$$

for some half-integer j. But

$$\chi(\theta) = -\chi_j(\theta) \implies I(\alpha, D_j) = -I(D_j, D_j) = -1,$$

which is impossible. Hence

$$\chi(\theta) = \chi_j(\theta),$$

and so (since a representation is determined up to equivalence by its character)

$$\alpha = D_j.$$

Finally, we show that

$$D_j D_k = D_{j+k} + D_{j+k-1} + \cdots + D_{|j-k|}.$$

It is sufficient to prove the corresponding result for the characters

$$\chi_j(\theta)\chi_k(\theta) = \chi_{j+k}(\theta) + \chi_{j+k-1}(\theta) + \cdots + \chi_{|j-k|}(\theta).$$

We may suppose that $j \geq k$. We prove the result by induction on k.
If $k = 0$ the result is trivial, since $\chi_0(\theta) = 1$. If $k = 1/2$ then

$$\begin{aligned}\chi_j(\theta)\chi_{1/2}(\theta) &= \left(e^{2ji\theta} + e^{2(j-1)i\theta} + e^{-2ji\theta}\right)\left(e^{i\theta} + e^{-i\theta}\right)\\ &= \left(e^{(2j+1)i\theta} + e^{-(2j-1)i\theta}\right) + \left(e^{(2j-1)i\theta} + e^{-(2j+1)i\theta}\right)\\ &= \chi_{j+1/2}(\theta) + \chi_{j-1/2}(\theta),\end{aligned}$$

as required.
Suppose $k \geq 1$. Then

$$\chi_k(\theta) = \chi_{k-1}(\theta) + (e^{ki\theta} + e^{-ki\theta}).$$

Thus applying our inductive hypothesis,

$$\chi_j(\theta)\chi_k(\theta) = \chi_{j+k-1}(\theta) + \cdots + \chi_{j-k+1} + \chi_j(\theta)(e^{ki\theta} + e^{-ki\theta}).$$

But

$$\begin{aligned}\chi_j(\theta)(e^{ki\theta} + e^{-ki\theta}) &= \left(e^{2ji\theta} + e^{2(j-1)i\theta} + e^{-2ji\theta}\right)\left(e^{ki\theta} + e^{-ki\theta}\right)\\ &= \chi_{j+k}(\theta) + \chi j - k(\theta),\end{aligned}$$

giving the required result

$$\begin{aligned}\chi_j(\theta)\chi_k(\theta) &= \chi_{j+k-1}(\theta) + \cdots + \chi_{j-k+1} + \chi_{j+k}(\theta) + \chi j - k(\theta)\\ &= \chi_{j+k}(\theta) + \cdots + \chi_{j-k}.\end{aligned}$$

4. Show that there exists a surjective homomorphism

$$\Theta : \mathbf{SU}(2) \to \mathbf{SO}(3)$$

with finite kernel.

Hence or otherwise determine all simple representations of $\mathbf{SO}(3)$.

Determine also all simple representations of $\mathbf{O}(3)$.

Answer: *The set of skew-hermitian* 2×2 *matrices*

$$S = \begin{pmatrix} ia & -b+ic \\ b+ic & id \end{pmatrix} \qquad (a, b, c, d \in \mathbb{R})$$

forms a 4-dimensional real vector space U*. The group* $\mathbf{SU}(2)$ *acts on this space by*

$$(U, S) \mapsto U^{-1}SU = U^*SU,$$

since

$$(U^*SU)^* = U^*S^*U = -U^*SU.$$

The 3-dimensional subspace $W \subset U$ *formed by trace-free skew-hermitian matrices*

$$T = \begin{pmatrix} ix & -y+iz \\ y+iz & -ix \end{pmatrix} \qquad (x, y, z \in \mathbb{R})$$

is stable under $\mathbf{SU}(2)$ *since*

$$\operatorname{tr}(U^*TU) = \operatorname{tr}(U^{-1}TU) = \operatorname{tr} T = 0.$$

Thus W *carries a representation of* $\mathbf{SU}(2)$ *of degree 3, corresponding to a homomorphism*

$$\Theta : \mathbf{SU}(2) \to \mathbf{GL}(3, \mathbb{R}).$$

Moreover, this homomorphism preserves the positive-definite quadratic form

$$\det T = x^2 + y^2 + z^2$$

on W *since*

$$\det(U^*TU) = \det(U^{-1}TU) = \det T.$$

Hence

$$\operatorname{im} \Theta \subset \mathbf{O}(3).$$

Finally, $\mathbf{SU}(2) \cong S^3$ *is connected; and so therefore is its image. But* $\mathbf{SO}(3)$ *is an open subgroup of* $\mathbf{O}(3)$*. Hence*

$$\operatorname{im} \Theta \subset \mathbf{SO}(3).$$

Thus our homomorphism takes the form

$$\Theta : \mathbf{SU}(2) \to \mathbf{SO}(3).$$

It remains to show that Θ *has a finite kernel, and is surjective.*
If

$$U \in \ker \Theta$$

then

$$U^{-1}TU = T$$

for all $T \in W$*. Each* $S \in U$ *can be expressed in the form*

$$S = T + \rho I,$$

where $T \in W$ *and* $\rho = \operatorname{tr} S/2$*. It follows that*

$$U^{-1}SU = S$$

for all skew-hermitian $S \in U$*.*
Hence

$$U^{-1}HU = H$$

for all hermitian H*, since* H *is hermitian if and only if* $S = iH$ *is skew-hermitian.*
It follows from this that

$$U^{-1}XU = X$$

for all $X \in \mathbf{Mat}(2, \mathbb{C})$*, since every* X *is expressible in the form*

$$X = H + S,$$

with $H = (X + X^*)/2$ *hermitian and* $S = (X - X^*)/2$ *skew-hermitian.*
But it is a simple matter to see that the only such U *are* $U = \pm I$*. Thus*

$$\ker \Theta = \{\pm I\}.$$

To see that Θ *is surjective, we note that if*

$$U(\theta) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

then

$$U(\theta)^{-1} \begin{pmatrix} ix & -y + iz \\ y + iz & -ix \end{pmatrix} U(\theta) = \begin{pmatrix} ix & e^{-2i\theta}(-y + iz) \\ e^{2i\theta}(y + iz) & -ix \end{pmatrix},$$

ie

$$\Theta U(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\theta & -\sin 2\theta \\ 0 & \sin 2\theta & \cos 2\theta \end{pmatrix} = R(Ox, 2\theta),$$

rotation about Ox through angle 2θ. In particular, $\operatorname{im}\Theta$ *contains all rotations about Ox.*

Now let

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Then $U \in \mathbf{SU}(2)$ and

$$U^{-1} \begin{pmatrix} ix & -y+iz \\ y+iz & -ix \end{pmatrix} U = \begin{pmatrix} iz & -y-ix \\ y-ix & -iz \end{pmatrix}.$$

Thus

$$\alpha(U) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = R(Oy, \pi/2).$$

Writing $P = R(Oy, \pi/2)$,

$$P^{-1} R(Ox, \theta) P = R(Oz, \theta).$$

Thus $\operatorname{im}\Theta$ *contains all rotations about Oz as well as Ox. But is is easy to see that every rotation $R \in \mathbf{SO}(3)$ is expressible as a product of rotations about Ox and Oz. Hence*

$$\operatorname{im}\Theta = \mathbf{SO}(3),$$

ie Θ is surjective.

Thus

$$\mathbf{SO}(3) = \mathbf{SU}(2)/\{\pm i\}.$$

It follows that the representations of $\mathbf{SO}(3)$ are just the representations α of $\mathbf{SO}(2)$ such that

$$\alpha(-I) = I.$$

In particular, the simple representations of $\mathbf{SO}(3)$ are those simple representations D_j of $\mathbf{SU}(2)$ such that $D_j(-I) = I$. But D_j is defined by the action of $\mathbf{SU}(2)$ on the polynomials

$$P(z, w) = c_0 z^{2j} + c_1 z^{2j-1} w + \cdots + c_{2j} w^{2j}.$$

It is clear that

$$P(-z, -w) = P(z, w)$$

for all P of degree $2j$ if and only if $2j$ is even, ie j is an integer.

Thus the simple representations of $\mathbf{SO}(3)$ *are* $D_0, D_1, D_2, \ldots$ *of degrees* $1, 3, 5, \ldots$.

Since

$$\mathbf{O}(3) = \mathbf{SO}(3) \times C_2,$$

where $C_2 = \{\pm I\}$*, the simple representations of* $\mathbf{O}(3)$ *are of the form* $\alpha \times \beta$*, where* α *is a simple representation of* $\mathbf{SO}(3)$*, and* β *is a simple representation of* C_2*. Thus the simple representations of* $\mathbf{O}(3)$ *are* $D_j \times 1$ *and* $D_j \times \epsilon$*, where* $j \in \mathbb{N}$ *and* ϵ *is the representation* $-I \to -1$ *of* C_2*.*

5. Explain the division of simple representations of a finite or compact group G over $\mathbb{C}$ into *real*, *essentially complex* and *quaternionic*. Give an example of each (justifying your answers).

   Show that if α is a simple representation with character χ then the value of

   $$\int_G \chi(g^2)\, dg$$

   determines which of these three types α falls into.

   Answer: *Suppose* α *is a simple representation of* G *over* $\mathbb{C}$*. Then* α *is said to be* real *if*

   $$\alpha = \beta_{\mathbb{C}}$$

   for some representation of G *over* $\mathbb{R}$*. If this is so then the character*

   $$\chi_\alpha(g) = \chi_\beta(g)$$

   is real. We say that α *is* quaternionic *if its character is real, but it is not real. Finally, we say that* α *is* essentially complex *if its character is not real.*

   The trivial character 1 *of any group is real, since it is the complexification of the trivial character over* $\mathbb{R}$*.*

   The 1-dimensional character θ *of the cyclic group* $C_3 = \langle g \rangle$ *given by*

   $$\theta : g \mapsto \omega = e^{2\pi/3}$$

   is essentially complex, since its character θ *is not real.*

   Consider the quaternion group

   $$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}.$$

   We can regard the quaternions $\mathbb{H}$ *as a 2-dimensional vector space over* $\mathbb{C} = \langle 1, i \rangle$*. The action of* Q_8 *on* $\mathbb{H}$ *by multiplication on the left defines a 2-dimensional representation* α *of* D_8*. We assert that this is a simple quaternionic representation.*

It is certainly simple, since otherwise $\mathbb{H}$ would have a 1-dimensional subspace $\langle q \rangle$ stable under D_8, and therefore under $\mathbb{H}$, since D_8 spans $\mathbb{H}$. But that is impossible since

$$x = (xq^{-1})q$$

for any $x \in \mathbb{H}$. The simple representations of D_4 must have dimensions $1, 1, 1, 1, 2$ (since $\sum \dim_i^2 = 8$). It follows that

$$\alpha^* = \alpha$$

since there is only 1 2-dimensional simple representation. Hence χ_α is real.

It remains to show that α is not real. Consider the 4-dimensional representation β of D_8 over $\mathbb{R}$, defined by the same action of D_4 on $\mathbb{H}$. This is easily seen to be simple, by the argument above. It follows that $\beta_{\mathbb{C}}$ is either simple, or splits into 2 simple representations over $\mathbb{C}$ of dimension 2. The only possibility is that

$$\beta_{\mathbb{C}} = 2\alpha.$$

Now if α were real, say

$$\alpha = \gamma_{\mathbb{C}}$$

we would deduce that $\beta = 2\gamma$ which is impossible, since β is simple.

Now suppose α is a simple representation of G in V. Then $(\alpha^*)^2$ is the representation arising from the action of G on the space of bilinear forms on V.

But

$$\alpha^* = \alpha \iff \chi_\alpha \text{ is real.}$$

Thus

$$I(1, (\alpha^*)^2) = I(\alpha, \alpha^*) = \begin{cases} 1 & \text{if } \alpha \text{ is real or quaternionic} \\ 0 & \text{if } \alpha \text{ is essentially complex} \end{cases}.$$

In other words, there is just 1 invariant bilinear form (up to a scalar multiple) if α is real or quaternionic, and no such form if α is essentially complex.

Now the space of bilinear forms splits into the direct sum of symmetric (or quadratic) and skew-symmetric forms, since each bilinear form $B(u, v)$ can be expressed as

$$B(u, v) = \frac{1}{2}\left(B(u, v) + B(v, u)\right) + \frac{1}{2}\left(B(u, v) - B(v, u)\right),$$

where the first form is symmetric and the second skew-symmetric.

It follows that

$$(\alpha^*)^2 = \phi + \psi,$$

where ϕ is the representation of G in the space of symmetric forms, and ψ the representation in the space of skew-symmetric forms.

If α is essentially complex, there is no invariant symmetric or skew-symmetric form. But if α is real or quaternionic, there must be just 1 invariant form, either symmetric or skew-symmetric. We shall see that in fact there is an invariant symmetric form if and only if α is real.

Certainly if α is real, say $\alpha = \beta_{\mathbb{C}}$, where β is a representation in the real vector space U, then we know that there is an invariant positive-definite form on U, and this will give an invariant quadratic form on $V = U_{\mathbb{C}}$.

Conversely, suppose α is a quaternionic simple representation on V. Then $\beta = \alpha_{\mathbb{R}}$ is simple. For

$$(\alpha_{\mathbb{R}})_{\mathbb{C}} = \alpha + \alpha^*$$

for any representation α over $\mathbb{C}$. Thus if $\beta = \gamma + \gamma'$ then (with α quaternionic)

$$2\alpha = \gamma_{\mathbb{C}} + \gamma'_{\mathbb{C}},$$

and it will follow that

$$\alpha = \gamma_{\mathbb{C}} = \gamma'_{\mathbb{C}},$$

so that α is real.

Since β is simple, there is a unique invariant quadratic form P on $V_{\mathbb{R}}$, and this form is positive-definite. But if there were an invariant quadratic form Q on V this would give an invariant quadratic form on $V_{\mathbb{R}}$, which would not *be positive-definite, since we would have*

$$Q(iu, iu) = -Q(u).$$

Thus if α is quaternionic, then there is no invariant quadratic form on V, and therefore there is an invariant skew-symmetric form.

It follows that we can determine which class α falls into by computing

$$I(1, \phi) \text{ and } I(1, \psi).$$

To this end we compute the characters of ϕ and ψ.

Suppose $g \in G$. Then we can diagonalise g, ie we can find a basis $e_1, \ldots, e_n$ of V consisting of eivenvectors, say

$$ge_i = \lambda_i e_i.$$

The space of quadratic forms is spanned by the $n(n+1)/2$ forms

$$x_i x_j \quad (i \leq j),$$

where $x_1, \ldots, x_n$ *are the coordinates with respect to the basis* $e_1, \ldots, e_n$. *It follows that*

$$\chi_\phi(g) = \sum \lambda_i^2 + \sum_{i<j} \lambda_i \lambda_j.$$

Now

$$\chi_\alpha(g) = \sum \lambda_i, \ \chi_\alpha(g^2) = \sum \lambda_i^2.$$

It follows that

$$\chi_phi(g) = \frac{1}{2} \left(\chi_\alpha(g)^2 + \chi_alpha(g^2) \right).$$

We deduce from this that

$$I(1, \phi) = \frac{1}{2\|G\|} \sum_{g \in G} \left(\chi_\alpha(g)^2 + \chi_\alpha(g^2) \right).$$

Since

$$\begin{aligned} I(1, \phi) + I(1, \psi) &= I(1, (\alpha^*)^2) \\ &= \frac{1}{\|G\|} \sum_g \chi_\alpha(g^{-1})^2 \\ &= \frac{1}{\|G\|} \sum_g \chi_\alpha(g)^2, \end{aligned}$$

it follows that

$$I(1, \psi) = \frac{1}{2\|G\|} \sum_{g \in G} \left(\chi_\alpha(g)^2 - \chi_\alpha(g^2) \right).$$

Putting all this together, we conclude that

$$\begin{aligned} \frac{1}{\|G\|} \sum_{g \in G} \chi_\alpha(g^2) &= I(1, \phi) - I(1, \psi) \\ &= \begin{cases} 1 & \text{if } \alpha \text{ is real,} \\ -1 & \text{if } \alpha \text{ is quaternionic,} \\ 0 & \text{if } \alpha \text{ is essentially complex.} \end{cases} \end{aligned}$$