Course 424

Group Representations III

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School of Mathematics Thursday, 8 May 1997 14:00–16:00

Answer as many questions as you can; all carry the same number of marks.
Unless otherwise stated, all representations are finite-dimensional over $\mathbb{C}$.

1. Define the *exponential* e^X of a square matrix X.

   Determine e^X in each of the following cases:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$X = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

   Which of these 6 matrices X are themselves expressible in the form $X = e^Y$, where Y is a real matrix? (Justify your answers in all cases.)

2. Define a *linear group*, and a *Lie algebra*; and define the Lie algebra $\mathscr{L}G$ of a linear group G, showing that it is indeed a Lie algebra.

   Determine the Lie algebras of $\mathbf{SO}(3)$ and $\mathbf{SU}(2)$, and show that they are isomomorphic.

   Are the groups themselves isomorphic?

3. Define the *dimension* of a linear group; and determine the dimensions of each of the following groups:

$$\mathbf{O}(n), \mathbf{SO}(n), \mathbf{U}(n), \mathbf{SU}(n), \mathbf{GL}(n, \mathbb{R}), \mathbf{SL}(n, \mathbb{R}), \mathbf{GL}(n, \mathbb{C}), \mathbf{SL}(n, \mathbb{C})?$$

4. Define a *representation* of a Lie algebra; and show how each representation α of a linear group G gives rise to a representation $\mathscr{L}\alpha$ of $\mathscr{L}G$.

   Determine the Lie algebra of $\mathbf{SL}(2, \mathbb{R})$; and show that this Lie algebra $\mathbf{sl}(2, \mathbb{R})$ has just 1 simple representation of each dimension $1, 2, 3, \ldots$.

5. Show that the only compact connected abelian linear groups are the tori $\mathbb{T}^n$.