Course 424

Group Representations Ia

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Seminar Room Wednesday, 29 January 1997 15:00–17:00

Attempt 6 questions. (If you attempt more, only the best 6 will be counted.) All questions carry the same number of marks. Unless otherwise stated, all groups are finite, and all representations are of finite degree over $\mathbb{C}$.

1. Define a *group representation*. What is meant by saying that the representation α is *simple*?

   Show that every simple representation of G is of degree $\leq |G|$.

   Determine all simple representations of the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ (up to equivalence) from first principles.

2. What is meant by saying that the representation α is *semisimple*?

   Prove that every representation α of a finite group G (of finite degree over $\mathbb{C}$) is semisimple.

   Define the *intertwining number* $I(\alpha, \beta)$ of 2 representations α, β.

   Show that the simple parts of a semisimple representation are unique up to order.

3. Define the *character* $\chi_\alpha(g)$ of a representation α.

   Explain how an action of a group G on a finite set X gives rise to a (permutation) representation α of G.

   Show that

   $$\chi_\alpha(g) = |\{x \in X : gx = x\}|.$$

   Determine the characters of S_4 defined by its actions on the set $X = \{a, b, c, d\}$ and the set Y consisting of the 6 subsets of X containing 2 elements.

   Hence or otherwise draw up the character table of S_4.

4. Show that if the simple representations of G are $\sigma_1, \ldots, \sigma_s$ then

   $$\dim^2 \sigma_1 + \cdots + \dim^2 \sigma_s = |G|.$$

   Determine the degrees of the simple representations of S_5.

5. Show that a simple representation of an abelian group is necessarily of degree 1.

   Prove conversely that if every simple representation of G is of degree 1 then G must be abelian.

   Show that the simple representations of an abelian group G themselves form a group (under multiplication) isomorphic to G.

6. Draw up the character table of the alternating group A_4.

   Determine also the *representation ring* of A_4, ie express each product of simple representations of A_4 as a sum of simple representations.

7. What is meant by saying that a simple representation α (over $\mathbb{C}$) is (a) *real*, (b) *quaternionic*?

   Show that if χ_α is real then α is real or quaternionic according as

   $$\frac{1}{|G|} \sum_{g \in G} \chi_\alpha(g^2) = \pm 1.$$

8. By considering the eigenvalues of 5-cycles, or otherwise, show that S_n has no simple representations of degree 2 or 3 if $n \geq 5$.