Course 424

Group Representations III

Dr Timothy Murphy

22 September 1987 Exam Hall 14:00–16:00

Answer as many questions as you can; all carry the same number of marks.
Unless otherwise stated, all Lie algebras are over $\mathbb{R}$, and all representations are finite-dimensional over $\mathbb{C}$.

1. Define a *representation* α of a Lie algebra, and explain carefully how a representation of a linear group G gives rise to a representation of its Lie algebra $\mathscr{L}G$.

   What is meant by saying that α is (a) *simple* (b) *semisimple*?

   Determine the simple representations of $\mathbf{sl}(2, \mathbb{R})$.

   Sketch the proof that all representations of $\mathbf{sl}(2, \mathbb{R})$ are semisimple.

2. Define the *adjoint representation* **ad** of a Lie algebra $\mathscr{L}$ (verifying that it is indeed a representation).

   Compute the adjoint representation of $\mathbf{so}(3)$ in matrix form.

   Define a *covering* $f : G \to H$, where G and H are connected linear groups. By using the adjoint representation **Ad** of the *group* $\mathbf{SU}(2)$, or otherwise, establish that there is a covering $f : \mathbf{SU}(2) \to \mathbf{SO}(3)$. What is $\ker f$?

3. Define the *Killing form* $K(X)$ of a Lie algebra.

   Compute the Killing forms of $\mathbf{su}(2)$ and $\mathbf{sl}(2, \mathbb{R})$.

   Show that the Killing form of a compact linear group G (or rather, of $\mathscr{L}G$) is negative-definite or negative-indefinite.

   Sketch the proof of the converse result, that if the Lie algebra $\mathscr{L}G$ of the connected linear group G is negative-definite then G is compact.

4. Show that any set of commuting matrices has a common eigenvector.

   Sketch the representation theory of $\mathbf{su}(3)$, defining carefully the terms *weight* and *weight vector*.

   Determine the weights of the adjoint representation of $\mathbf{su}(3)$.