Course 424

Group Representations IIa

Dr Timothy Murphy

Joly Theatre Tuesday, 17 April 2001 14:00–15:30

Attempt 5 questions. (If you attempt more, only the best 5 will be counted.) All questions carry the same number of marks. All representations are finite-dimensional over $\mathbb{C}$.

1. Define a *measure* on a compact space. State carefully, and outline the main steps in the proof of, Haar's Theorem on the existence of an invariant measure on a compact group.

2. Which of the following groups are (a) compact, (b) connected:
$$\mathrm{O}(n), \mathrm{SO}(n), \mathrm{U}(n), \mathrm{SU}(n), \mathrm{GL}(n,\mathbb{R}), \mathrm{SL}(n,\mathbb{R}), \mathrm{GL}(n,\mathbb{C}), \mathrm{SL}(n,\mathbb{C})?$$
(Justify your answer in each case.)

3. Prove that every simple representation of a compact abelian group is 1-dimensional and unitary.

   Determine the simple representations of $\mathrm{SO}(2)$.

   Determine also the simple representations of $\mathrm{O}(2)$.

4. Determine the conjugacy classes in $\mathrm{SU}(2)$. Prove that $\mathrm{SU}(2)$ has just one simple representation of each dimension $m = 1, 2, 3, \ldots$; and determine the character of this representation.

5. Show that there exists a surjective homomorphism
$$\Theta : \mathrm{SU}(2) \to \mathrm{SO}(3)$$
with finite kernel.

   Hence or otherwise determine all simple representations of $\mathrm{SO}(3)$.

6. Show that every function $f(g)$ on a finite group G is expressible as a linear combination of the functions
$$\chi(ag)$$
as χ runs through the simple characters of G, and a runs through the elements of G.