Course 424

Group Representations II

Dr Timothy Murphy

Joly Theatre Friday, 6 April 2001 16:00–17:30

Attempt 5 questions. (If you attempt more, only the best 5 will be counted.) All questions carry the same number of marks. All representations are finite-dimensional over $\mathbb{C}$*.*

1. Define a *measure* on a compact space. State carefully, and outline the main steps in the proof of, Haar's Theorem on the existence of an invariant measure on a compact group.

2. Which of the following groups are (a) compact, (b) connected:

   $$\mathrm{O}(n), \mathrm{SO}(n), \mathrm{U}(n), \mathrm{SU}(n), \mathrm{GL}(n, \mathbb{R}), \mathrm{SL}(n, \mathbb{R}), \mathrm{GL}(n, \mathbb{C}), \mathrm{SL}(n, \mathbb{C})?$$

   (Justify your answer in each case.)

3. Prove that every representation of a compact group is semisimple.

   Give an example of a representation of the additive group $\mathbb{R}$ which is not semisimple.

4. Prove that every simple representation of a compact *abelian* group is 1-dimensional.

   Determine the simple representations of $\mathrm{U}(1)$.

5. Determine the conjugacy classes in $\mathrm{SU}(2)$.

   Prove that $\mathrm{SU}(2)$ has just one simple representation of each dimension $1, 2, \ldots$; and determine the character of this representation.

6. Determine the conjugacy classes in $\mathrm{SO}(3)$.

   Prove that $\mathrm{SO}(3)$ has just one simple representation of each odd dimension $1, 3, 5, \ldots$.