Course 424

Group Representations I

Dr Timothy Murphy

Seminar Room Friday, 9 February 2001 16:00–18:00

Attempt 6 questions. (If you attempt more, only the best 6 will be counted.) All questions carry the same number of marks. Unless otherwise stated, all groups are finite, and all representations are of finite degree over $\mathbb{C}$.

1. Define a *group representation*. What is meant by saying that 2 representations α, β are *equivalent*? Determine all representations of S_3 of degree 2 (up to equivalence) from first principles.

   What is meant by saying that the representation α is *simple*? Determine all simple representations of S_3 from first principles.

2. What is meant by saying that the representation α is *semisimple*?

   Prove that every representation α of a finite group G (of finite degree over $\mathbb{C}$) is semisimple.

   Show that the natural n-dimensional representation of S_n in C^n (by permutation of coordinates) is the sum of 2 simple representations.

3. Define the *character* χ_α of a representation α.

   Define the *intertwining number* $I(\alpha, \beta)$ of 2 representations α, β. State and prove a formula expressing $I(\alpha, \beta)$ in terms of χ_α, χ_β.

   Show that the simple parts of a semisimple representation are unique up to order.

4. Show that the number of simple representations of a finite group G is equal to the number of conjugacy classes in G.

5. Show that a finite group G has only a finite number of simple representations (up to equivalence), say $\sigma_1, \dots, \sigma_r$; and show that

$$(\deg \sigma_1)^2 + \cdots + (\deg \sigma_r)^2 = \|G\|.$$

Show that the number of simple representations of S_n of degree d is even if d is odd. Hence or otherwise determine the dimensions of the simple representations of S_5.

6. Draw up the character table of S_4.

Determine also the *representation ring* of S_4, ie express each product of simple representations of S_4 as a sum of simple representations.

7. Define the representation $\alpha \times \beta$ of the product-group $G \times H$, where α is a representation of G, and β of H.

Show that $\alpha \times \beta$ is simple if and only if both α and β are simple; and show that every simple representation of $G \times H$ is of this form.

Show that D_6 (the symmetry group of a regular hexagon) is expressible as a product group

$$D_6 = C_2 \times S_3.$$

Let γ denote the 3-dimensional representation of D_6 defined by its action on the 3 diagonals of the hexagon. Express γ in the form

$$\gamma = \alpha_1 \times \beta_1 + \cdots + \alpha_r \times \beta_r,$$

where $\alpha_1, \dots, \alpha_r$ are simple representations of C_2, and $\beta_1, \dots, \beta_r$ are simple representations of S_3.

8. Explain the division of simple representations (over $\mathbb{C}$) into *real*, *essentially complex* and *quaternionic*. Give an example of each (justifying your answers).

Show that if α is a simple representation with character χ then the value of

$$\sum_{g \in G} \chi(g^2)$$

determines which of these 3 types α falls into.