



Course 424

Group Representations IIb

Dr Timothy Murphy

Mathematics 1.8 Friday, 16 April 1999 16:00–17:30

Answer as many questions as you can; all carry the same number of marks.

All representations are finite-dimensional over $\mathbb{C}$.

1. What is meant by a *measure* on a compact space X ? What is meant by saying that a measure on a compact group G is *invariant*? Sketch the proof that every compact group G carries such a measure.

Show that every representation of a compact group is semisimple.

2. Which of the following groups are (a) compact, (b) connected:

$$\mathbf{O}(n), \mathbf{SO}(n), \mathbf{U}(n), \mathbf{SU}(n), \mathbf{GL}(n, \mathbb{R}), \mathbf{SL}(n, \mathbb{R}).$$

Justify your answer in each case; no marks will be given for unsupported assertions.

3. Determine the conjugacy classes in $\mathbf{SU}(2)$. Prove that $\mathbf{SU}(2)$ has just one simple representation of each dimension $1, 2, 3, \dots$; and determine the character of this representation.

4. Show that there exists a surjective homomorphism

$$\Theta : \mathbf{SU}(2) \rightarrow \mathbf{SO}(3)$$

with finite kernel.

Hence or otherwise determine all simple representations of $\mathbf{SO}(3)$.

5. Define the exterior product $\lambda^r \alpha$ of a representation α .

Show that the character of $\lambda^2 \alpha$ is given by

$$\chi_{\lambda^2 \alpha}(g) = \frac{1}{2} (\chi_{\alpha}(g)^2 - \chi_{\alpha}(g^2)).$$

Hence or otherwise show that if $D(j)$ is the representation of $\mathbf{SU}(2)$ of dimension $2j + 1$ then

$$\lambda^2 D(j) = D(2j - 1) + D(2j - 3) + \cdots + \begin{cases} D(1) & \text{if } j \text{ is integral} \\ D(0) & \text{if } j \text{ is half-integral} \end{cases}.$$