Course 424

Group Representations Ia

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Seminar Room Thursday, 17 June 1999 16:30–18:30

Attempt 6 questions. (If you attempt more, only the best 6 will be counted.) All questions carry the same number of marks. Unless otherwise stated, all groups are finite, and all representations are of finite degree over $\mathbb{C}$.

1. Define a *group representation*. What is meant by saying that 2 representations α, β are *equivalent*? What is meant by saying that the representation α is *simple*?

   Determine all simple representations of S_3 up to equivalence, from first principles.

2. What is meant by saying that the representation α is *semisimple*?

   Prove that every representation α of a finite group G (of finite degree over $\mathbb{C}$) is semisimple.

3. Define the *intertwining number* $I(\alpha, \beta)$ of 2 representations α, β.

   Show that if α, β are simple then

$$I(\alpha, \beta) = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta. \end{cases}$$

   Hence or otherwise show that the simple parts of a semisimple representation are unique up to order.

4. Define the *character* χ_α of a representation α.

   State without proof a formula expressing the intertwining number $I(\alpha, \beta)$ in terms of the characters χ_α, χ_β.

   Show that two representations α, β of G are equivalent if and only if they have the same character.

5. Draw up the character table of S_4.

6. Draw up the character table of D_5 (the symmetry group of a regular pentagon).

   Determine also the *representation ring* of D_5, ie express each product of simple representations of D_5 as a sum of simple representations.

7. Show that the number of simple representations of a finite group G is equal to the number s of conjugacy classes in G.

8. Show that if the simple representations of the finite group G are $\sigma_1, \ldots, \sigma_s$ then

   $$(\deg \sigma_1)^2 + \cdots + (\deg \sigma_s)^2 = \|G\|.$$

   Show that the number of simple representations of S_n of degree d is even if d is odd. Hence or otherwise determine the dimensions of the simple representations of S_5.