Course 424

Group Representations I

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Exam Hall Monday, 13 January 1997 14:00–16:00

Attempt 6 questions. (If you attempt more, only the best 6 will be counted.) All questions carry the same number of marks. Unless otherwise stated, all groups are finite, and all representations are of finite degree over $\mathbb{C}$.

1. Define a *group representation.* What is meant by saying that 2 representations α, β are *equivalent*? Determine all representations of S_3 of degree 2 (up to equivalence) from first principles.

   What is meant by saying that the representation α is *simple*? Determine all simple representations of S_3 from first principles.

2. What is meant by saying that the representation α is *semisimple*?

   Prove that every representation α of a finite group G (of finite degree over $\mathbb{C}$) is semisimple.

   Show that the natural n-dimensional representation of S_n in C^n (by permutation of coordinates) is the sum of 2 simple representations.

3. Define the *character* χ_α of a representation α.

   Define the *intertwining number* $I(\alpha, \beta)$ of 2 representations α, β. State and prove a formula expressing $I(\alpha, \beta)$ in terms of χ_α, χ_β.

   Show that the simple parts of a semisimple representation are unique up to order.

4. Explain how a representation β of a subgroup $H \subset G$ *induces* a representation β^G of G.

   Show that

$$\frac{\bar{g}}{|G|}\chi_{\beta^G}(\bar{g}) = \sum_{\bar{h}\subset\bar{g}} \frac{\bar{h}}{|H|}\chi_{\beta}(\bar{h}).$$

   Determine the characters of S_4 induced by the simple characters of S_3, and hence or otherwise draw up the character table of S_4.

5. Show that the number of simple representations of a finite group G is equal to the number s of conjugacy classes in G.

   Show also that if these representations are $\sigma_1, \ldots, \sigma_s$ then

$$\dim^2 \sigma_1 + \cdots + \dim^2 \sigma_s = |G|.$$

6. Draw up the character table of the dihedral group D_5 (the symmetry group of a regular pentagon).

   Determine also the *representation ring* of D_5, ie express each product of simple representations of D_5 as a sum of simple representations.

7. Define the representation $\alpha \times \beta$ of the product-group $G \times H$, where α is a representation of G, and β of H.

   Show that $\alpha \times \beta$ is simple if and only if both α and β are simple; and show that every simple representation of $G \times H$ is of this form.

   Show that D_6 (the symmetry group of a regular hexagon) is expressible as a product group

$$D_6 = C_2 \times S_3.$$

   Let γ denote the 3-dimensional representation of D_6 defined by its action on the 3 diagonals of the hexagon. Express γ in the form

$$\gamma = \alpha_1 \times \beta_1 + \cdots + \alpha_r \times \beta_r,$$

   where $\alpha_1, \ldots, \alpha_r$ are simple representations of C_2, and $\beta_1, \ldots, \beta_r$ are simple representations of S_3.

8. By considering the eigenvalues of 5-cycles, or otherwise, show that S_n has no simple representation of degree 2 if $n > 4$.