Course 424

Group Representations II

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G.M.B. Tuesday, 13 April 1993 14:00–16:00

Answer as many questions as you can; all carry the same number of marks.
Unless otherwise stated, all representations are finite-dimensional over $\mathbb{C}$.

1. Define a *measure* on a compact space. State carefully, and outline the main steps in the proof of, Haar's Theorem on the existence of an invariant measure on a compact group.

   Prove that every representation of a compact group is semisimple.

2. Which of the following groups are (a) compact, (b) connected:

   $\mathbf{SO}(2), \mathbf{O}(2), \mathbf{U}(2), \mathbf{SU}(2), \mathbf{Sp}(2), \mathbf{GL}(2, \mathbb{R}), \mathbf{SL}(2, \mathbb{R}), \mathbf{GL}(2, \mathbb{C}), \mathbf{SL}(2, \mathbb{C}), \mathbb{T}^2$?

   (Note: Justify your answer in each case; no marks will be given for unsupported assertions)

3. Determine the conjugacy classes in $\mathbf{SU}(2)$; and prove that this group has just one simple representation of each dimension.

   Find the character of the representation $D(j)$ of dimensions $2j + 1$ (where $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$).

   Express each product $D(i)D(j)$ as a sum of simple representations $D(k)$.

4. Show that there exists a surjective homomorphism

   $$\Theta : \mathbf{SU}(2) \to \mathbf{SO}(3)$$

   with finite kernel.

   Hence or otherwise determine all simple representations of $\mathbf{SO}(3)$.

   Determine also all simple representations of $\mathbf{O}(3)$.

5. Explain the division of simple representations of a finite or compact group G into *real*, *essentially complex* and *quaternionic*.

   Determine into which of these 3 categories the representations $D(j)$ of $\mathbf{SU}(2)$ fall.