Course 424

Group Representations I

Dr Timothy Murphy

G.M.B. Friday, 22 January 1993 14:00–16:00

Answer as many questions as you can; all carry the same number of marks.
Unless otherwise stated, all groups are finite, and all representations are finite-dimensional over $\mathbb{C}$.

1. Define a *group representation*. What is meant by saying that 2 representations α, β are *equivalent*? Determine all 2-dimensional representations of S_3 (up to equivalence) from first principles.

   What is meant by saying that the representation α is *simple*? Determine all simple representations of S_3 from first principles.

2. What is meant by saying that the representation α is *semisimple*?

   Prove that every finite-dimensional representation α of a finite group over $\mathbb{C}$ is semisimple.

   Prove also that the simple parts of a semisimple representation are unique up to order.

   Show that the natural n-dimensional representation of S_n in C^n (by permutation of coordinates) is the sum of 2 simple representations.

3. Define the *character* χ_α of a representation α, and show that it is a class function (constant on conjugacy classes).

   Define the *intertwining number* $I(\alpha, \beta)$ of 2 representations α, β, and show that
$$I(\alpha, \beta) = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_\alpha(g)} \chi_\beta(g).$$

   Prove that a representation α is simple if and only if $I(\alpha, \alpha) = 1$.

   Prove that every simple representation of an abelian group is 1-dimensional.

4. Explain how a representation β of a subgroup $H \subset G$ *induces* a representation β^G of G.

   Show that
$$\frac{\bar{g}}{|G|} \chi_{\beta^G}(\bar{g}) = \sum_{\bar{h} \subset \bar{g}} \frac{\bar{h}}{|H|} \chi_\beta(\bar{h}).$$

   Determine the characters of S_4 induced by each of the simple characters of S_3, and so draw up the character table of S_4.

5. Show that the number of simple representations of a finite group G is equal to the number s of conjugacy classes in G.

   Show also that if these representations are $\sigma_1, \ldots, \sigma_s$ then
$$\dim^2 \sigma_1 + \cdots + \dim^2 \sigma_s = |G|.$$

   Determine the dimensions of the simple representations of S_5, stating clearly any results you assume.

6. Define the representation $\alpha \times \beta$ of the product-group $G \times H$, where α is a representation of G, and β of H.

   Show that $\alpha \times \beta$ is simple if and only if both α and β are simple; and show that every simple representation of $G \times H$ is of this form.

   Show that D_6 (the symmetry group of a regular hexagon) is expressible as a product group
$$D_6 = C_2 \times S_3.$$

   Let γ denote the 6-dimensional representation of D_6 defined by its action on the 6 vertices of the hexagon. Express γ in the form
$$\gamma = \alpha_1 \times \beta_1 + \cdots + \alpha_r \times \beta_r,$$

   where $\alpha_1, \ldots, \alpha_r$ are simple representations of C_2, and $\beta_1, \ldots, \alpha_r$ are simple representations of S_3.