# Course 424

# Group Representations I

Dr Timothy Murphy

Exam Hall Friday, 8 March 1991 15:00–17:00

*Answer as many questions as you can; all carry the same number of marks.*
*Unless otherwise stated, all groups are finite, and all representations are finite-dimensional over $\mathbb{C}$.*

1. Define a *group representation.* What is meant by saying that 2 representations $\alpha, \beta$ are *equivalent*? What is meant by saying that the representation $\alpha$ is *simple*?

   Determine all simple representations of $S_3$ up to equivalence, from first principles.

2. What is meant by saying that the representation $\alpha$ is *semisimple*?

   Prove that every finite-dimensional representation $\alpha$ of a finite group over $\mathbb{C}$ is semisimple.

   Define the *character* $\chi_\alpha$ of a representation $\alpha$.

   Define the *intertwining number* $I(\alpha, \beta)$ of 2 representations $\alpha, \beta$. State and prove a formula expressing $I(\alpha, \beta)$ in terms of $\chi_\alpha, \chi_\beta$.

   Show that the simple parts of a semisimple representation are unique up to order.

3. Explain how a representation $\beta$ of a subgroup $H \subset G$ *induces* a representation $\beta^G$ of $G$.

   Show that
$$\frac{\bar{g}}{|G|}\chi_{\beta^G}(\bar{g}) = \sum_{\bar{h}\subset\bar{g}} \frac{\bar{h}}{|H|}\chi_\beta(\bar{h}).$$

   Determine the characters of $S_4$ induced by the simple characters of $S_3$, and so draw up the character table of $S_4$.

4. Show that the number of simple representations of a finite group $G$ is equal to the number $s$ of conjugacy classes in $G$.

   Show also that if these representations are $\sigma_1, \ldots, \sigma_s$ then
$$\dim^2 \sigma_1 + \cdots + \dim^2 \sigma_s = |G|.$$

   Determine the dimensions of the simple representations of $S_5$, stating clearly any results you assume.

5. Draw up the character table of the alternating group $A_4$ (the subgroup of $S_4$ formed by even permutations).

   Determine also the *representation ring* of $A_4$, ie express each product of simple representations of $A_4$ as a sum of simple representations.

6. Explain the division of simple representations (over $\mathbb{C}$) into *real*, *essentially complex* and *quaternionic*. Give an example of each (justifying your answers).

   Show that if $\alpha$ is a simple representation with character $\chi$ then the value of
$$\sum_{g\in G} \chi(g^2)$$
   determines which of these 3 types $\alpha$ falls into.