Course 424

Group Representations III

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Joly Theatre Friday, 25 April 2003 15:15–16:45

Attempt 5 questions. (If you attempt more, only the best 5 will be counted.) All questions carry the same number of marks. In this exam, 'Lie algebra' means Lie algebra over $\mathbb{R}$, *and 'representation' means* finite-dimensional representation over $\mathbb{C}$.

1. Define the *exponential* e^X of a square matrix X, and show that e^X is invertible.

   Determine e^X in each of the following cases:

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$
$$X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad X = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

   Which of these 6 matrices X are themselves expressible in the form $X = e^Y$, (a) with a real matrix Y, (b) with a complex matrix Y? (Justify your answers in all cases.)

2. Define a *Lie algebra*.

   Show that there are just two Lie algebras over $\mathbb{R}$, up to isomorphism; and give two linear groups having these as their Lie algebras.

Continued overleaf

3. Define the *dimension* of a linear group; and determine the dimensions of each of the following groups:

$$\mathrm{O}(n), \mathrm{SO}(n), \mathrm{U}(n), \mathrm{SU}(n), \mathrm{GL}(n, \mathbb{R}),$$
$$\mathrm{SL}(n, \mathbb{R}), \mathrm{GL}(n, \mathbb{C}), \mathrm{SL}(n, \mathbb{C}), \mathrm{Sp}(n), \mathrm{E}(n)?$$

   ($\mathrm{E}(n)$ is the isometry group of n-dimensional Euclidean space.)

4. Determine the Lie algebras of $\mathrm{SU}(2)$ and $\mathrm{SO}(3)$, and show that they are isomomorphic.

   Show that the 2 groups themselves are *not* isomorphic.

5. Determine the Lie algebra of $\mathrm{SL}(2, \mathbb{R})$, and find all the simple representations of this algebra.

   Do these representations all arise from representations of the group $\mathrm{SL}(2, \mathbb{R})$?

6. What is meant by saying that a topological group G is *linearisable*? Show that the Lie algebra $\mathscr{L}G$ is well-defined in such a case.

   Show that the group $\mathrm{E}(3)$ (the group of isometries of Euclidean space) is linearisable, and determine its Lie algebra $\mathrm{e}(3)$.

7. Show that every connected abelian linear group A is isomorphic to

$$\mathbb{T}^m \times \mathbb{R}^n$$

   for some m and n, where $\mathbb{T}$ denotes the torus $\mathbb{R}/\mathbb{Z}$.

   Express the group $\mathbb{C}^\times$ in this way.

   Show that the groups $\mathbb{T}^m \times \mathbb{R}^n$ and $\mathbb{T}^{m'} \times \mathbb{R}^{n'}$ are isomorphic if and only if $m = m'$ and $n = n'$.