Course 424

Group Representations IIa

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Room WR3.8 Wednesday, 30 April 2003 15:15–16:45

Attempt 5 questions. (If you attempt more, only the best 5 will be counted.) All questions carry the same number of marks. All representations are finite-dimensional over $\mathbb{C}$, unless otherwise stated.

1. What is meant by a *measure* on a compact space X? What is meant by saying that a measure on a compact group G is *invariant*? Sketch the proof that every compact group G carries such a measure. To what extent is this measure unique?

2. Prove that every simple representation of a compact abelian group is 1-dimensional and unitary.

   Determine the simple representations of $\mathrm{SO}(2)$.

   Determine also the simple representations of $\mathrm{O}(2)$.

3. Determine the conjugacy classes in $\mathrm{SU}(2)$; and prove that this group has just one simple representation of each dimension.

   Find the character of the representation $D(j)$ of dimensions $2j + 1$ (where $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$).

   Determine the representation-ring of $\mathrm{SU}(2)$, ie express each product $D(i)D(j)$ as a sum of simple representations $D(k)$.

4. Show that there exists a surjective homomorphism

$$\Theta : \mathrm{SU}(2) \to \mathrm{SO}(3)$$

with finite kernel.

Hence or otherwise determine all simple representations of SO(3).

Determine also all simple representations of O(3).

5. Explain the division of simple representations of a finite or compact group G over $\mathbb{C}$ into *real*, *essentially complex* and *quaternionic*. Give an example of each (justifying your answers).

Show that if α is a simple representation with character χ then the value of

$$\int_G \chi(g^2)\, dg$$

determines which of these three types α falls into.

6. Define the representation $\alpha \times \beta$ of the product-group $G \times H$, where α is a representation of G, and β of H.

Show that if G and H are finite then $\alpha \times \beta$ is simple if and only if both α and β are simple; and show that every simple representation of $G \times H$ is of this form.

Show that the symmetry group G of a cube is expressible as a product-group

$$G = C_2 \times S_4.$$

Let γ denote the 4-dimensional representation of G defined by its action on the 4 diagonals of the cube. Express γ in the form

$$\gamma = \alpha_1 \times \beta_1 + \cdots + \alpha_r \times \beta_r,$$

where $\alpha_1, \ldots, \alpha_r$ are simple representations of C_2, and $\beta_1, \ldots, \beta_r$ are simple representations of S_4.