Course 424

Group Representations I

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Walton Theatre   Friday, 10 January 2003   16:00–18:00

Attempt 6 questions. (If you attempt more, only the best 6 will be counted.) All questions carry the same number of marks. Unless otherwise stated, all groups are finite, and all representations are of finite degree over $\mathbb{C}$.

1. Define a *group representation.* What is meant by saying that 2 representations α, β are *equivalent*?

   Determine all 2-dimensional representations of D_5 up to equivalence, from first principles.

2. What is meant by saying that the representation α is *simple*?

   Determine all simple representations of D_4, from first principles.

3. What is meant by saying that the representation α is *semisimple*?

   Prove that every finite-dimensional representation α of a finite group over $\mathbb{C}$ is semisimple.

   Show that the natural representation of S_n in $\mathbb{C}^n$ (by permutation of coordinates) splits into 2 simple parts, for any $n > 1$.

Continued overleaf

4. Define the *character* χ_α of a representation α.

   Define the *intertwining number* $I(\alpha, \beta)$ of 2 representations α, β. State and prove a formula expressing $I(\alpha, \beta)$ in terms of χ_α, χ_β.

   Show that the simple parts of a semisimple representation are unique up to order.

5. Draw up the character table of S_4, explaining your reasoning throughout.

   Determine also the *representation ring* of S_4, ie express each product of simple representations of S_4 as a sum of simple representations.

6. Explain how a representation β of a subgroup $H \subset G$ *induces* a representation β^G of G.

   State (without proof) a formula for the character of β^G in terms of that of β.

   Determine the characters of S_4 induced by the simple characters of S_3, expressing each induced character as a sum of simple parts.

7. Determine the character table of the quaternion group

$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}.$$

8. Show that the number of simple representations of a finite group G is equal to the number of conjugacy classes in G.