Course 424

Group Representations III

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Joly Theatre   Friday, 4 May 2001   14:00–15:30

Attempt 5 questions. (If you attempt more, only the best 5 will be counted.) All questions carry the same number of marks. In this exam, 'Lie algebra' means Lie algebra over $\mathbb{R}$, *and 'representation' means* finite-dimensional representation over $\mathbb{C}$.

1. Define the *exponential* e^X of a square matrix X.

   Determine e^X in each of the following cases:

$$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$
$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

   Which of these 6 matrices X are themselves expressible in the form $X = e^Y$, where Y is a real matrix? (Justify your answers in all cases.)

2. Define a *linear group*, and a *Lie algebra*; and define the Lie algebra $\mathscr{L}G$ of a linear group G, showing that it is indeed a Lie algebra.

   Show that a homomorphims of linear groups $f : G \to H$ gives rise to a Lie algebra homomorphism $\mathscr{L}f : \mathscr{L}G \to \mathscr{L}H$

   If f is surjective, does it necessarily follow that $\mathscr{L}f$ is surjective? If f is injective, does it necessarily follow that $\mathscr{L}f$ is injective? (Give reasons.)

Continued overleaf

3. Define the *dimension* of a linear group; and determine the dimensions of each of the following groups:

$$\mathrm{O}(n), \mathrm{SO}(n), \mathrm{U}(n), \mathrm{SU}(n), \mathrm{GL}(n, \mathbb{R}),$$
$$\mathrm{SL}(n, \mathbb{R}), \mathrm{GL}(n, \mathbb{C}), \mathrm{SL}(n, \mathbb{C}), \{\mathrm{Sp}(n), E(n)?$$

($E(n)$ is the isometry group of n-dimensional Euclidean space.)

4. Determine the Lie algebras of SU(2) and SO(3), and show that they are isomomorphic.

   Show that the 2 groups themselves are *not* isomorphic.

5. Determine the Lie algebra of $\mathrm{SL}(2, \mathbb{R})$, and find all the simple representations of this algebra.

   Show that every representation of the group $\mathrm{SL}(2, \mathbb{R})$ is semisimple, stating carefully but without proof any results you need.

6. Show that every connected abelian linear group A is isomorphic to

$$\mathbb{T}^m \times \mathbb{R}^n$$

   for some m and n, where $\mathbb{T}$ denotes the torus $\mathbb{R}/\mathbb{Z}$.

   Show that the groups $\mathbb{T}^m \times \mathbb{R}^n$ and $\mathbb{T}^{m'} \times \mathbb{R}^{n'}$ are isomorphic if and only if $m = m'$ and $n = n'$.