Course 424

Group Representations I

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Joly Theatre   Friday, 19 January 2001   16:00–18:00

Attempt 6 questions. (If you attempt more, only the best 6 will be counted.) All questions carry the same number of marks. Unless otherwise stated, all groups are finite, and all representations are of finite degree over $\mathbb{C}$.

1. Define a *group representation*. What is meant by saying that 2 representations α, β are *equivalent*?

   Determine all 2-dimensional representations of D_4 up to equivalence, from first principles.

2. What is meant by saying that the representation α is *simple*?

   Determine all simple representations of S_3, from first principles.

3. What is meant by saying that the representation α is *semisimple*?

   Prove that every finite-dimensional representation α of a finite group over $\mathbb{C}$ is semisimple.

   Show that the natural representation of S_n in $\mathbb{C}^n$ (by permutation of coordinates) splits into 2 simple parts, for any $n > 1$.

4. Define the *character* χ_α of a representation α.

   Define the *intertwining number* $I(\alpha, \beta)$ of 2 representations α, β. State and prove a formula expressing $I(\alpha, \beta)$ in terms of χ_α, χ_β.

   Show that the simple parts of a semisimple representation are unique up to order.

5. Draw up the character table of S_4, explaining your reasoning throughout.

   Determine also the *representation ring* of S_4, ie express each product of simple representations of S_4 as a sum of simple representations.

6. Explain how a representation β of a subgroup $H \subset G$ *induces* a representation β^G of G.

   State (without proof) a formula for the character of β^G in terms of that of β.

   Determine the characters of S_4 induced by the simple characters of the Viergruppe V_4, expressing each induced character as a sum of simple parts.

7. Define the representation $\alpha \times \beta$ of the product-group $G \times H$, where α is a representation of G, and β of H.

   Show that $\alpha \times \beta$ is simple if and only if both α and β are simple; and show that every simple representation of $G \times H$ is of this form.

   Show that D_6 (the symmetry group of a regular hexagon) is expressible as a product group
   $$D_6 = C_2 \times S_3.$$

   Let γ denote the 3-dimensional representation of D_6 defined by its action on the 3 diagonals of the hexagon. Express γ in the form
   $$\gamma = \alpha_1 \times \beta_1 + \cdots + \alpha_r \times \beta_r,$$
   where $\alpha_1, \ldots, \alpha_r$ are simple representations of C_2, and $\beta_1, \ldots, \beta_r$ are simple representations of S_3.

8. Show that a finite group G has only a finite number of simple representations (up to equivalence), say $\sigma_1, \ldots, \sigma_r$; and show that
   $$(\deg \sigma_1)^2 + \cdots + (\deg \sigma_r)^2 = \|G\|.$$

   Show that the number of simple representations of S_n of degree d is even if d is odd. Hence or otherwise determine the dimensions of the simple representations of S_5.