MA1M01 Calculus Assignment 8 Solutions

Michælmas term week 10

www.maths.tcd.ie/pub/MA1M01/Calculus/

1. **[40 points]** Carbon–14 (^{14}C) is a radioactive isotope of carbon and has a half-life of $t_{\frac{1}{2}} \approx 5730$ years. It undergoes beta decay via ${}^{14}_{6}\mathrm{C} \to {}^{14}_{7}\mathrm{N} + e^- + \bar{\nu}_e$. Given that this decay is exponential, it can be modelled as

$$N(t) = ce^{-\lambda t}$$

where $N(t)$ is the number of atoms of ^{14}C at time t (measured in years).

(a) What is the physical meaning of the constant c?

Let $t = 0$. Then $N(0) = ce^0 = c$, so $c = N(0)$ is the number of ^{14}C atoms that we begin with.

(b) Find the exponential decay rate λ.

We know that $N(t) = N(0)e^{-\lambda t}$ and $N(t_{\frac{1}{2}}) = \frac{1}{2}N(0)$. Putting these together,

$$\begin{aligned} \tfrac{1}{2}N(0) &= N(0)e^{-\lambda t_{\frac{1}{2}}}, \\ \tfrac{1}{2} &= e^{-\lambda t_{\frac{1}{2}}}, \end{aligned}$$

so $\ln(\frac{1}{2}) = -\lambda t_{\frac{1}{2}}$, giving

$$t_{\frac{1}{2}} = \frac{\ln(2)}{\lambda}, \qquad \begin{aligned} \lambda &= \frac{\ln(2)}{t_{\frac{1}{2}}} \\ &= \frac{\ln(2)}{5730} \\ &= 1.2097 \times 10^{-4}. \end{aligned}$$

(c) You find yourself in Hamunaptra (the lost city of the dead) and discover canopic jars carrying Anck–su–Namun's preserved organs. If $\frac{1}{3}$ of the ^{14}C is gone, how old are these organs?

$$N(t_{\frac{2}{3}}) = \tfrac{2}{3}N(0) = N(0)e^{-\lambda t_{\frac{2}{3}}},$$

so

$$\begin{aligned} \ln(\tfrac{2}{3}) &= -\lambda t_{\frac{2}{3}}, \\ t_{\frac{2}{3}} &= -\frac{\ln(\frac{2}{3})}{\lambda} \\ &\approx 3351.8, \end{aligned}$$

which seems about right.

(d) A band of American treasure hunters want to sell you the *Book of the Dead*. If 0.5% of the ^{14}C is lost, how old is it?

This is the same as the last question.

$$t_{0.995} = -\frac{\ln(0.995)}{\lambda} = 41.437,$$

so it is likely a forgery.

2. [**40 points**] Newton's law of cooling states that

$$\frac{d}{dt}T(t) = -k(T(t) - T_a),$$

where $T(t)$ is the temperature of the object at time t (measured in hours), T_a is the ambient temperature and k is some constant of proportionality.

(a) Show that $T(t) = ce^{-kt} + T_a$ is a solution to the above equation.

To show this, you just have to differentiate $T = ce^{-kt} + T_a$ and demonstrate that it satisfies Newton's law of cooling above.

$$\begin{aligned} T' &= \frac{d}{dt}(ce^{-kt} + T_a) \\ &= c\frac{d}{dt}e^{-kt} \\ &= -kce^{-kt} \\ &= -k(T - T_a) \end{aligned}$$

as $e^{-kt} = T - T_a$. This is enough to answer the question. But if you're curious, here's a derivation.

$$\begin{aligned} \int \frac{dT}{T - T_a} &= -k \int dt \\ &= -kt + c \end{aligned}$$

and letting $u = T - T_a$ gives

$$\begin{aligned} \int \frac{dT}{T - T_a} &= \int \frac{1}{u}\,du \\ &= \ln(u) + c \\ &= \ln(T - T_a) + c, \end{aligned}$$

so

$$\ln(T - T_a) = -kt + c$$

and exponentiating gives

$$\begin{aligned} T - T_a &= e^{-kt+c} \\ &= e^{-kt}e^{c} \\ &= ce^{-kt}, \\ T &= ce^{-kt} + T_a \end{aligned}$$

where I renamed e^c as c on the third line.

(b) What is the physical meaning of the constant c?

Again, let $t = 0$ to arrive at $T(0) = c + T_a$. Thus $c = T(0) - T_a$ is the difference between the initial temperature and the ambient temperature.

(c) You find the body of High Priest Imhotep at time t_0 after his death at a temperature of $T(t_0) = 26°C$. One hour later, his temperature is measured to be $T(t_0 + 1) = 25°C$. If the ambient temperature of the room is 18°C, what is the value of the constant k?

At t_0,

$$T(t_0) = 26 = T_a + ce^{-kt_0},$$

therefore

$$e^{-kt_0} = \frac{26 - T_a}{c}.$$

At $t_0 + 1$,

$$\begin{aligned} T(t_0 + 1) = 25 &= T_a + ce^{-k(t_0+1)} \\ &= T_a + ce^{-kt_0}e^{-k} \\ &= T_a + c\left(\frac{26 - T_a}{c}\right)e^{-k} \\ &= T_a + (26 - T_a)e^{-k}, \end{aligned}$$

$$\begin{aligned} e^{-k} &= \frac{25 - T_a}{26 - T_a}, \\ k = -\ln &\left(\frac{25 - T_a}{26 - T_a}\right) \approx 0.133. \end{aligned}$$

(d) How long has Imhotep been dead for? I.e. what is the value of t_0? Assume that Imhotep has a normal body temperature of 37°C.

From

$$\begin{aligned} e^{-kt_0} &= \frac{26 - T_a}{c}, \\ -kt_0 &= \ln\left(\frac{26 - T_a}{c}\right) = \ln\left(\frac{8}{19}\right), \end{aligned}$$

so

$$t_0 = -\frac{1}{k}\ln\left(\frac{8}{19}\right) \approx 6.5037.$$