MA1M01 Calculus Assignment 4 Solutions

Michælmas term week 6

www.maths.tcd.ie/pub/MA1M01/Calculus/

1. **[40 points]** Differentiate the following functions with respect to x.

   (a) $f(x) = (2x+7)^{10}$

   $f'(x) = 10(2x+7)^9 \cdot 2 = 20(2x+7)^9$.

   (b) $g(x) = \sqrt{x+5}$

   $g'(x) = \frac{1}{2}(x+5)^{-\frac{1}{2}} \cdot 1 = \frac{1}{\sqrt{x+5}}$.

   (c) $h(x) = \frac{2}{\sqrt{x}} + 8x^3$

   $h'(x) = 2\frac{-1}{2}x^{-\frac{3}{2}} + 24x^2 = -x^{-\frac{3}{2}} + 24x^2$.

   (d) $m(x) = f(g(x-5) - \frac{7}{2})$

   $m(x) = f(\sqrt{x} - \frac{7}{2}) = \left(2(\sqrt{x} - \frac{7}{2}) + 7\right)^{10} = (2\sqrt{x})^{10} = 2^{10}x^5$,

   $m'(x) = 2^{10} \cdot 5x^4 = 5120x^4$.

2. **[20 points]**

   (a) Compute $\int (17x+a)^8\,dx$ (where a is some number).
   Let $u = 17x + a$. Then $du = 17\,dx$ and

$$\begin{aligned}\int (17x+a)^8\,dx &= \frac{1}{17}\int u^8\,du \\ &= \frac{1}{17}\cdot\frac{1}{9}u^9 + c \\ &= \frac{1}{153}(17x+a)^9 + c.\end{aligned}$$

   (b) Find a function $f(x)$ such that $f'(x) = x\sqrt{x^2+4}$ and $f(0) = 0$.

$$f(x) = \int f'(x)\,dx = \int x\sqrt{x^2+4}\,dx.$$

   Let $u = x^2 + 4$. Then $du = 2x\,dx$ and

$$\begin{aligned}\int f'(x)\,dx &= \frac{1}{2}\int \sqrt{u}\,du \\ &= \frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}} + c \\ &= \frac{1}{3}(x^2+4)^{\frac{3}{2}} + c.\end{aligned}$$

But $f(0) = 0$ implies

$$\frac{1}{3}(0+4)^{\frac{3}{2}} + c = 0,$$

so

$$c = -\frac{4^{\frac{3}{2}}}{3} = -\frac{8}{3}.$$

Then

$$f(x) = \frac{1}{3}(x^2+1)^{\frac{3}{2}} - \frac{8}{3}.$$

This is defined for all $x \in \mathbb{R}$ and maps to all non–negative real numbers, so

$$f : \mathbb{R} \to \mathbb{R}^+ \cup \{0\}.$$

3. **[15 points]** Compute the following integrals.

   (a) $\int_{-2}^{6} dx = [x]_{-2}^{6} = 6 + 2 = 8.$

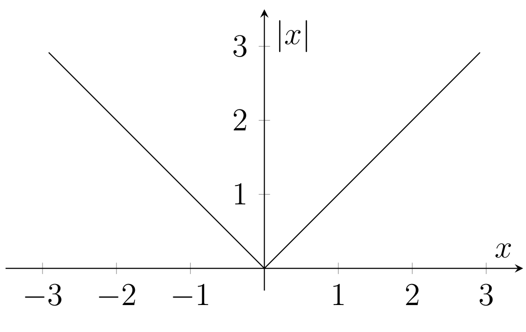
   (b) $\int_{0}^{3} x\,dx = \left[\frac{x^2}{2}\right]_0^3 = \frac{9}{2}.$

   (c) $\int_{-3}^{0} -x\,dx = \left[-\frac{x^2}{2}\right]_{-3}^{0} = 0 + \frac{(-3)^2}{2} = \frac{9}{2}.$

4. **[15 points]** The absolute value function is defined as

$$|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}.$$

   (a) Graph $|x|$ for $-3 < x < 3$.



   (b) Calculate $\int_{-3}^{3} |x|\,dx$.

$$\begin{aligned} \int_{-3}^{3} |x|\,dx &= \int_{-3}^{0} |x|\,dx + \int_{0}^{3} |x|\,dx \\ &= \int_{-3}^{0} -x\,dx + \int_{0}^{3} x\,dx \\ &= \frac{9}{2} + \frac{9}{2} \\ &= 9. \end{aligned}$$

5. [**10 points**] What is the area under the curve of

$$f(x) = x^3\sqrt{x^4+1}$$

between $x = -1$ and $x = 2$?

$$\begin{aligned}\int x^3\sqrt{x^4+1}\,dx &= \int x^3 u^{\frac{1}{2}}\,dx \\ &= \frac{1}{4}\int u^{\frac{1}{2}}\,du \\ &= \frac{1}{4}\cdot\frac{2}{3}u^{\frac{3}{2}} + c \\ &= \frac{1}{6}(x^4+1)^{\frac{3}{2}} + c,\end{aligned}$$

so

$$\begin{aligned}\int_{-1}^{2} x^3\sqrt{x^4+1}\,dx &= \left[\frac{1}{6}(x^4+1)^{\frac{3}{2}}\right]_{-1}^{2} \\ &= \frac{1}{6}\Big((2^4+1)^{\frac{3}{2}} - ((-1)^4+1)^{\frac{3}{2}}\Big) \\ &= \frac{1}{6}\Big(17^{\frac{3}{2}} - 2^{\frac{3}{2}}\Big) \\ &\approx 11.211.\end{aligned}$$