A VINDICATION OF SIR ISAAC NEWTON'S PRINCIPLES OF FLUXIONS, AGAINST THE OBJECTIONS CONTAINED IN THE ANALYST

By

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NOTE ON THE TEXT

This pamphlet has been transcribed from a copy in the library of Trinity College, Dublin.

The following spellings, differing from modern British English, are employed in the original 1735 edition: knowlege, expresly, lead [led], persue.

David R. Wilkins
Dublin, June 2002

A VINDICATION OF Sir *ISAAC NEWTON*'S Principles of FLUXIONS, AGAINST THE OBJECTIONS Contained in the ANALYST.

By *J. WALTON.*

—Si quid novisti rectius istis
Candidus imperti: Si non, his utere mecum. Hor.

In the Fulness of his Sufficiency he shall be in Straits: Every Hand of the WICKED *shall come upon him.* JOB.

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A

VINDICATION, *&c.*

UNDER Pretence of some Abuses committed by Mathematicians, in Virtue of the Authority they derive from their Profession, the Author of the *Minute Philosopher*, in a *Libel* called the *Analyst*, has declared 'em Infidels, Makers of Infidels, and Seducers of Mankind in Matters of the highest Concernment: This he professes to have done, not from any real Knowlege of his own, but from the credible Information of others; but he has neither produc'd his Informers, nor proved the Accusation in any one Instance; and therefore it is Defamatory.

But they assume an Authority, it seems, in Things foreign to their Profession, and undertake to decide in Matters whereof their Knowledge can by no Means qualify them to be competent Judges: And as this Practice, if not prevented, may be of dangerous Consequence; he has undertaken to enquire into the Object, Principles and Method of Demonstration, admitted by the Mathematicians of the present Age, with the same Freedom, he says, they presume to treat the Principles and Mysteries of Religion, to the end, that all Men may see what Right they have to lead or what Encouragement others have to follow them.

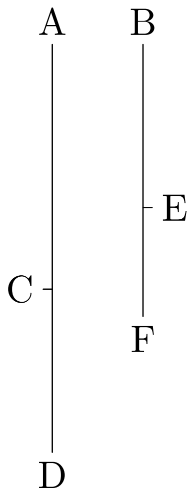
And whereas Sir *Isaac Newton* has presum'd to interpose in *Prophecies* and *Revelations*, and to decide in religious Affairs, it has been thought proper to begin with his Method of *Fluxions*, and to try what cou'd be done with that Method, with the Inventor himself, and with his Followers: And what has been done with 'em every intelligent Reader is able to judge.

If this Writer may be credited, the Objects about which the Method of *Fluxions* is conversant, are difficult to conceive or imagine distinctly; the Notions are most abstracted incomprehensible Metaphysics, not to be admitted for the Foundations of clear and accurate Science; the Principles are obscure, repugnant, precarious, the Arguments admitted in Proofs, are fallacious, indirect, illogical; and the Inferences and Conclusions not more just, than the Conceptions of the Principles are clear.

How far the Credulous and Injudicious may become infected by this uncommon Way of treating Mathematics and Mathematicians, is not easy to foresee, and therefore it will be necessary to give a short Account of the Nature of *Fluxions*, and of the Objects about which the Method is conversant; and when it shall be made apparent, that this Author has not understood the Metaphysics he wou'd refute; it will not be difficult to defend the Principles and their Demonstrations, from any Imputations of Fallacy or Repugnancy, which have yet been pointed at by him or any other Writer.

"In the Method of *Fluxions*, Sir *Isaac Newton* considers mathematical Quantities, not as composed of the smallest Parts, but as described or generated by continual Motion. Lines are described, and by being described are generated, not by an Apposition of Parts, but by the Motion of Points; Surfaces by the Motion of Lines, Solids by the Motion of Surfaces, Angles by the Rotation of their Sides, Times by a continual Flux, and so of the rest. And by considering that Quantities, increasing in equal Times, and generated by increasing, become greater or less, according as the Velocity with which they increase, and are generated, is greater or less, he found a Method of determining the Quantities themselves from the Velocities of the Motions, or of the Increments, with which they are generated; calling the Velocities of the Motions or of the Increments *Fluxions*, and the Quantities generated *Fluents*."

The momentaneous Increments or Decrements of flowing Quantities, he elsewhere calls by the Name of *Moments*, and considers the Increments as added or affirmative Moments, and the Decrements as subducted or negative ones: By Moments we may understand the nascent or evanescent Elements or Principles of finite Magnitudes, but not Particles of any determinate Size, or Increments actually generated; for all such are Quantities themselves, generated of Moments.



The Magnitudes of the momentaneous Increments or Decrements of Quantities are not regarded in the Method of Fluxions, but their first or last Proportions only: that is, the Proportions with which they begin or cease to exist: These are not their Proportions immediately before or after they begin or cease to exist, but the Proportions with which they begin to exist, or with which they vanish. If the Lines A C and B E are supposed to be generated in the same Time, by the Motions of the Points A and B, to C and E; and if by continuing the Motions of those Points to D and F, they generate D C and E F, synchronal Increments of A C and B E; it is evident that the Points D and F may flow back in the same Time to C and E, and by flowing back perpetually lessen the Magnitudes of those Increments till at last they vanish together, when the Points D and F come to coincide with C and E: Now the ultimate Ratio of those Increments is that Ratio with which they vanish and become nothing; or the Ratio with which they cease to be: And the first Ratio of them is the Ratio with which they begin to exist, at the very first setting out of the Points from C to E towards D and F.

Hence, if the describing Points move back to C and E, in the same Time wherein by moving forward they generated the Increments D C and E F; and in returning have every where the same Velocities, at certain Distances from C and E, which they had at those Distances in going forward; the last and first Ratios of the Increments will be equal, or they will vanish, and become nothing, with the very same Ratio with which they began to exist.

Hence likewise it appears, that to obtain the last Ratio of synchronal Increments, the Magnitudes of those Increments must be infinitely diminish'd. For their last Ratio is the Ratio with which they vanish or become nothing: But they cannot vanish or become nothing, by a constant Diminution, till they are infinitely diminish'd; for without an infinite Diminution they must have finite or assignable Magnitudes, and while they have finite or assignable Magnitudes they cannot vanish.

The ultimate Ratios with which synchronal Increments of Quantities vanish, are not the Ratios of finite Increments, but Limits which the Ratios of those Increments attain, by having their Magnitudes infinitely diminish'd: The Proportions of Quantities which grow less and less by Motion, and at last cease to be, will in most cases continually change, and become different in every successive Diminution of the Quantities themselves: And there are certain determinate Limits to which all such Proportions perpetually tend, and approach nearer than by any assignable Difference, but never attain before the Quantities themselves are infinitely diminish'd; or till the Instant they evanesce and become nothing. These Limits are the last Ratios with which such Quantities or their Increments vanish or cease to exist; and they are the first Ratios with which Quantities or the Increments of Quantities, begin to arise or come into being.

Quantities, and the Ratios of Quantities, which constantly tend to an Equality, by a Diminution of their Difference, and before the End of some finite Time approach nearer to an Equality than by any assignable Difference, at last become equal. For they become equal when the Difference between them vanishes or becomes nothing; and it will vanish or become nothing by being infinitely diminished: If the Quantities A C and A D perpetually tend to an Equality, either by the Motion of the Point D to C, or by that of C to D; they will become equal, and their Ratio a Ratio of Equality, when their Difference C D, by a constant Diminution, vanishes and becomes nothing, which it will do under a Coincidence of the two Points in C or D; and then either A D becomes A C, and so $\frac{AD}{AC}$ or $\frac{AC}{AC}$ is a Ratio of Equality, or else A C becomes A D and $\frac{AD}{AC}$ becomes $\frac{AD}{AD}$; which is also a Ratio of Equality.

The Fluxions of Quantities are very nearly as the Increments of their Fluents generated in the least equal Particles of Time: If C D and E F be Increments of the Fluents A C and B E, described in the least equal Particles of Time; the Fluxions in the Points C and E will be nearly as the Increments D C and E F. For from the exceeding Smallness of the Times it is evident that the Points D and F, must be extreamly near to C and E; and by Consequence however the Velocities are accelerated or retarded thro' the Spaces C D and E F, they will be very nearly the same in D and F as they were in C and E: But Velocities which are very nearly uniform, will be very nearly proportional to the Spaces described by them in equal Times; and therefore the Velocities in the Points C and E, which are the Fluxions of A C and B E in those Points, will be very nearly as the Increments D C and E F, described in the least equal Particles of Time.

The Fluxions of Quantities are accurately in the first or last Proportions of their nascent or evanescent Increments: Thus the Fluxions of A C and B E, in the Points C and E, are in the first or last Ratio of the Increments C D and E F. For the first or last Ratio of the Increments C D and E F, is the Ratio with which they begin or cease to exist: But the Ratio with which they begin or cease to exist, is the same with the Ratio of the Velocities in C and E, which are the Fluxions in those Points; and consequently the Fluxions in C and E are in the first or last Ratio of the Increments C D and E F.

The Fluxions of Quantities are only the Velocities with which those Quantities begin to be generated or increased; or the Velocities with which the generating Quantities begin to set out; not the Velocities they have after moving thro' Spaces of any finite or assignable Magnitudes: And therefore if two mathematical Quantities set out together, and begin to move with Velocities which are as a and b, they must begin to describe Spaces in the same Proportion with a and b; or the Proportion with which the Spaces begin to exist or to be described, must be the same with that which the Velocities have at the very Beginning of the Motion. For in the very Beginning of the Motion there is neither any Change of Velocity from Acceleration or Retardation, nor Difference of Time.

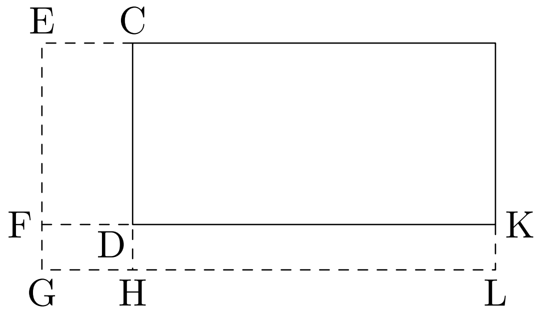
Hence it appears that to obtain the Ratios of Fluxions, the corresponding synchronal or isochronal Increments must be lessened *in infinitum*. For the Magnitudes of synchronal or isochronal Increments must be infinitely diminished and become evanescent, in order to obtain their first or last Ratios, to which Ratios the Ratios of their corresponding Fluxions are equal.

Hence likewise it appears that the Moments of like Quantities, compared with each other, are in Ratios compounded of the Ratios of the generating Quantities, taken when first they begin to move, and of the Velocities with which they set out: Or in Ratios compounded of the Ratios of the generating Quantities when first they begin to move, and of the first Ratios of the Spaces described by them in equal Times. The Moments of Lines therefore are as the generating Points and as the Velocities with which they begin to move taken together: The Moments of Surfaces, which become greater or less by carrying of moveable Lines along immoveable ones, are in Ratios compounded of the Ratios of the moving Lines, and of their first Velocities, or first Ratios of the Increments which begin to rise with those Velocities: And the whole Motion by which Squares or Rectangles begin to alter, either from an Augmentation or Diminution of their Sides, is the Sum of the nascent Motions of those Sides, or the Sum of the nascent Increments arising with the first Motions of the Sides: For the Proportion of nascent Increments is the same with that of the Motions with which they begin to be generated.

From this short Account of the Nature of Fluxions, compared with the *Analyst*, it appears that the Author of that Paper is greatly mistaken in the Object of 'em; and he is also mistaken in the Principles: For he thinks the Moment or Fluxion of a Rectangle, contain'd under two indeterminate Quantities A and B, from whence are deduc'd Rules for obtaining the Moments or Fluxions of all other Products or Powers whatever, is no where truly determin'd by Sir *Isaac Newton*: But he ought to have read Sir *Isaac* with more Care and Attention than he seems to have done, before he set up to decide and dictate in Matters of this Nature; and he wou'd do well yet to read him with Attention.

If any Rectangle C K be increased from an Augmentation of its Sides by Motion, so as that D K becomes L G in the same Time as D C becomes E G; the Moment of that Rectangle is the Sum of the Rectangles of D K into the Moment of D C, and of D C into the Moment of D K: That is, putting A and B for the Sides D K and D C, and a and b for their respective Moments, the Moment of the Rectangle A B will be $\mathrm{A}\,b + \mathrm{B}\,a$.

For the Gnomon C G K in the Instant it begins or ceases to exist is the Moment of the Rectangle C K: But the first or last Ratio of that Gnomon to the Sum of the Rectangles L D and F C is a Ratio of Equality: For the Difference between the Gnomon and the Sum of those Rectangles perpetually lessens, by a constant Diminution of the Increments F D and D H, or



by an approach of the Points F and H towards D; as will be manifest on taking the Ratio between the said Gnomon and the Sum of the Rectangles, at several Distances of the Points F and H from D: For whatever be the Magnitudes of a and b, when F and H first begin to move back towards D, the Gnomon C G K and Sum of the Rectangles L D and F C, will be as $\mathrm{A}\,b+\mathrm{B}\,a+b\,a$ and $\mathrm{A}\,b+\mathrm{B}\,a$; when those Points, by moving towards D, have lessen'd the Increments of D K and D C to $\frac{1}{2}\,a$ and $\frac{1}{2}\,b$, the Gnomon and Sum of the Rectangles will be as $\mathrm{A}\,b+\mathrm{B}\,a+\frac{1}{2}\,b\,a$ and $\mathrm{A}\,b+\mathrm{B}\,a$; when they have lessen'd the increments to $\frac{1}{4}\,a$ and $\frac{1}{4}\,b$, the Gnomon and Sum of the Rectangles will be as $\mathrm{A}\,b+\mathrm{B}\,a+\frac{1}{4}a\,b$ and $\mathrm{A}\,b+\mathrm{B}\,a$; and as $\mathrm{A}\,b+\mathrm{B}\,a+\frac{1}{8}a\,b$ and $\mathrm{A}\,b+\mathrm{B}\,a$, when they have lessen'd those Increments to $\frac{1}{8}\,a$ and $\frac{1}{8}\,b$: Hence it appears, that under a constant Diminution of the Increments a and b, by the Motion of the Points F and H towards D, the Gnomon C G K and the Sum of the Rectangles C F and D L, constantly tend to an Equality by a continual Diminution of their Difference F H, and that they become equal, and their Ratio becomes a Ratio of Equality, in the Instant that Difference vanishes and the Points F and H coincide with D; or in other Words the Gnomon and Sum of the Rectangles L D and F C begin or cease to *be* under a Ratio of Equality: And therefore the Sum of those Rectangles, or $\mathrm{A}\,b+\mathrm{B}\,a$, is the Moment of A B.

Hence, the Gnomon C D K, or $\mathrm{A}\,b+\mathrm{B}\,a+a\,b$, found by taking the Difference between the Rectangles E L and C K, or by deducting the Rectangle A B from a Rectangle contain'd under the Sides A and B increased by their whole Increments, is not the Moment or Fluxion of the Rectangle A B, except in the very Instant when it begins or ceases to exist: And this will also appear by considering it in another Light. For the Moment of the Rectangle C K, or the Motion with which it first begins to alter, either by increasing or decreasing, is the Sum of the nascent Motions of its Sides; and the nascent Motions of its Sides, are measur'd by their respective Magnitudes in the very Instant they first begin to change, and by the Velocities with which they begin to move taken together; and the Velocities with which the Sides begin to move being in the first Ratio of the momentaneous Spaces which arise with 'em; it follows that the Sum of the nascent Motions of the Sides, is the Sum of D K multiply'd into D H in its nascent State, and of C D multiply'd into D F in its nascent State: But D H and D F in their nascent States, are the Moments of D C and D K: And therefore the whole Moment of the Rectangle A B, is $\mathrm{A}\,b+\mathrm{B}\,a$.

In determining the Moments of Quantities, Sir *Isaac Newton* expresly tells us, that we are only to consider the Ratios with which they begin or cease to exist; and to obtain those Ratios, it is not necessary that the isochronal Increments shou'd have finite Magnitudes. "*Cave tamen intellexeris particulas finitas*," says he, "*Particulæ finitæ non sunt Momenta, sed Quantitates ipsæ ex Momentis genitæ. Intelligenda sunt Principia jamjam nascentia finitarum Magnitudinum. Neque enim spectatur in hoc Lemmate magnitudo Momentorum,* *sed prima nascentium proportio.*" And in another Place, "*Fluxiones sunt quam proximè ut Fluentium Augmenta æqualibus Temporis particulis quam minimis genita, et, ut accurate loquar, sunt in primâ ratione Augmentorum nascentium; exponi autem possunt per lineas quascunque, quæ sunt ipsis proportionales.*" And again, "*Siquando facili rerum conceptui consulens dixero Quantitates quam minimas, vel evanescentes, vel ultimas; cave intelligas quantitates magnitudine determinatas, sed cogita semper diminuendas sine limite.*"

From these Passages it appears, that the Gnomon C G K in its nascent or evanescent State only, or in the Instant it begins or ceases to exist, is the Moment or Fluxion of the Rectangle C K; and in a nascent or evanescent State, when only the Increments of Quantities become their Moments, its Ratio to $\mathrm{A}\,b+\mathrm{B}\,a$, which is the Sum of the Rectangles L D and F C, is a Ratio of Equality. By diminishing the Magnitudes of a and b, which are Increments of D K and D C, it is obvious that the Gnomon C G K diminishes faster in Proportion, than the Sum of the Rectangles F C and D L does; and by diminishing faster, it continually approaches to an Equality with that Sum, and attains the Equality only, when their Difference F H becomes evanescent, that is, when the Points F and H come to coincide with D; so that here is no Artifice of false Reasoning used, to get rid of H F or $a\,b$, that Term having no Existence at the very Beginning of the Motion, or in the nascent State of the Augments.

After Sir *Isaac* had so expresly told us what he meant by Moments and Fluxions, and by nascent or evanescent Quantities, one wou'd imagine it impossible to have mistaken and misrepresented him in the Manner this Author has done. He seems indeed to have been lead, or rather to have been deceived, by an Opinion that there can be no first or last Ratios of mathematical Quantities or of their isochronal Increments generated or destroy'd by Motion; imagining that no such Quantities, by any Division or Diminution whatever, can be exhausted or reduc'd to nothing: But if Lines, Surfaces and Solids can be generated or augmented by the Motion of Points, Lines and Surfaces, they may likewise be destroy'd or diminish'd by the Motion of the same Points, Lines and Surfaces, in returning to the Places from whence they first set out. While a generating Quantity moves back thro' the same Space it before described in moving forward, the Quantity generated, or its Augment, continually lessens; and by persevering in a State of decreasing, it must in some finite Time vanish and become nothing; and therefore mathematical Quantities, by a constant Diminution, may be reduc'd to nothing: And such as are thus generated or destroy'd in equal Times by Motion, or which arise and vanish together, will arise or vanish under certain Ratios, which are their first or last Ratios: or the Ratios with which they begin or cease to be: But it may be necessary to persue this Case a little farther, and see whether Sir *Isaac Newton*'s Demonstration of it cannot be defended, and proved to be geometrical.

"Suppose any Rectangle A B augmented by continual Motion; and the momentaneous Increments of its Sides A and B to be denoted by a and b, the Moment of the generated Rectangle will be measured by $\mathrm{A}\,b + \mathrm{B}\,a$.

"For when the Sides A and B wanted half of their Moments, the Rectangle was $\overline{\mathrm{A} - \frac{1}{2}\,a} \times \overline{\mathrm{B} - \frac{1}{2}\,b}$ or $\mathrm{A\,B} - \frac{1}{2}\,\mathrm{A}\,b - \frac{1}{2}\,\mathrm{B}\,a + \frac{1}{4}\,a\,b$: And as soon as the Sides A and B are augmented by the other halves of their Moments, it becomes $\overline{\mathrm{A} + \frac{1}{2}\,a} \times \overline{\mathrm{B} + \frac{1}{2}\,b}$ or $\mathrm{A\,B} + \frac{1}{2}\,\mathrm{A}\,b + \frac{1}{2}\,\mathrm{B}\,a + \frac{1}{4}\,a\,b$: From this Rectangle deduct the former, and there will remain $\mathrm{A}\,b + \mathrm{B}\,a$: Therefore the Increment of the Rectangle A B, generated with a and b the whole Increments of the Sides, is $\mathrm{A}\,b + \mathrm{B}\,a$."

Now, in determining the Moment of a Rectangle, there is nothing to be considered, when it first begins to be augmented by the Motions of its Sides, but the Sides themselves and the Velocities with which they begin to move; or the Sides and the first Ratio of the Spaces described by them. And therefore the true Moment of the Rectangle A B, or the Law according to which it begins to be augmented, on the Principles of Sir *Isaac Newton*, will only be the Sum of the Rectangles $\mathrm{A}\,b$ and $\mathrm{B}\,a$; for the Sides A and B begin to move with the Velocities which are as b and a: But this Moment $\mathrm{A}\,b + \mathrm{B}\,a$, is manifestly equal to the Difference between the Rectangles $\overline{\mathrm{A} + \frac{1}{2}\,a} \times \overline{\mathrm{B} + \frac{1}{2}\,b}$ and $\overline{\mathrm{A} - \frac{1}{2}\,a} \times \overline{\mathrm{B} - \frac{1}{2}\,b}$; and therefore Sir *Isaac*'s determination of it is geometrical.

From the foregoing Principle so demonstrated, the general Rule for finding the Moment or Fluxion of any Power of a flowing Quantity, is easily deduc'd: It is easy, from hence, to infer that the Moment or Fluxion of A^n is $n\mathrm{A}^{n-1}$, or that the Fluxion of x^n is as nx^{n-1}: But because this is also determined in a manner seemingly different, by Sir *Isaac*, in his Introduction to the *Quadrature of Curves*, the Author of the *Analyst* observes, "That there seems to have been some inward Scruple or Consciousness of Defect in the foregoing Demonstration." And he repeats the same Reflection in another Place, adding withal, "That Sir *Isaac* was not enough pleased with any one Notion steadily to adhere to it": But Reflections of this Nature deserve no Regard, unless it be allowable, by way of Return, to observe that the Person who makes 'em has very often been guilty of like Practices himself.*

The Proof given in the Introduction to the *Quadratures*, is said to be a most inconsistent way of arguing; as proceeding to a certain Point of the Demonstration upon Supposition of an Increment, and then in a fallacious Manner, shifting the Supposition to that of no Increment; and to shew the Inconsistency with greater Force, a Lemma is premised by Way of Axiom; as if some very obvious and natural Application of an apparent Truth, wou'd at one overturn the Whole of Sir *Isaac*'s Demonstration: But that Lemma, however true in it self, is no Way pertinent to the Case for which it was intended; and therefore such Inferences as are made in Virtue of it, with relation to the Point in dispute, are illegitimate, and inconsistent with the Rules of true reasoning.

Nothing is more plain and obvious, than that Quantities which begin to exist together under certain Proportions, and with certain Velocities; may become evanescent and cease to exist, under the same Proportions and with the same Velocities; and this is all Sir *Isaac* supposes in that determination of the Fluxion of x^n; and it is not very obvious, that the Lemma which this Author has hit upon, is applicable to Cases of such a Nature.

That the Reader may see how strictly Sir *Isaac Newton* has kept to the same Principle in this Determination, how steadily he adheres to the same Method, and how ill the Author of the *Analyst* has proved his Imputations; it will be necessary to persue this Point, and consider the Proof it self.

Let it be required to find the Fluxion of x^n, supposing x to increase uniformly.

Suppose x in any finite Particle of Time, to become greater than before, by a finite Increment, whose Magnitude is express'd by o. Then, in the same Time that x, by flowing becomes $x + o$, the n Power of x will become

$$x^n + nox^{n-1} + \frac{n^2 - n}{2}\,o^2x^{n-2} + \&c.$$

* See his new *Theory of Vision*; his *Treatise on the Principles of Human Knowlege*; and some later Undertakings of equal Importance.

Consequently the Magnitudes of the synchronal Increments of x and of x^n, are to each other as

$$1 \quad \text{and} \quad nx^{n-1} + \frac{n^2-n}{2}\,ox^{n-2} + \textit{\&c.}$$

Now, let the Increments decrease by flowing back, in like Manner as they increas'd before by flowing forward, and continually grow less and less till they vanish; and their ultimate Ratio, that is, the Ratio with which they become evanescent, will be express'd by 1 and nx^{n-1}: But the Fluxions of Quantities are in the last Ratio of their evanescent Auguments; and by Consequence the Fluxion of x is to that of x^n as 1 to nx^{n-1}.

In this Computation, Sir *Isaac* endeavours to collect the Proportion with which the isochronal Increments of x and of x^n, begin or cease to exist: Their Proportion obtain'd on Supposition that o is something, is allowed to be the same with that of

$$1 \quad \text{and} \quad nx^{n-1} + \frac{n^2-n}{2}\,ox^{n-2} + \textit{\&c.}$$

And it must be acknowleg'd that this Ratio has a Limite dependent on the Magnitude of o, which Limite it cannot attain before the Increments are infinitely diminish'd and become evanescent; and when, by an infinite Diminution, they become evanescent, no other Terms of their Ratio will be affected, so as to vanish with 'em, but such as are govern'd or regulated by them: In the Instant therefore that o vanishes, $\frac{n^2-n}{2}\,ox^{n-2}$ and all ensuing Terms of the Series absolutely vanish together; but the terms 1 and nx^{n-1} remain invariable under all possible Changes of the Increments, from any finite Degrees of Magnitude whatever, even till they become evanescent: They therefore express the last Ratio, under which the isochronal Increments of x and x^n vanish, or the Proportion of the Velocities with which those Increments cease to exist: Sir *Isaac Newton* then rightly retain'd 'em for the Measures of the Ratio of the Fluxions of x and x^n, tho' got in Virtue of his first Supposition; and the Fallacy, the Inconsistency, lies on the Side of this Author; who wou'd have them rejected on the Authority of a Lemma not to the purpose.

To make this Point still more plain and obvious, I shall propose the reasoning in a stronger Light: It amounts therefore to this, or may in other Words, be thus expressed: If x is suppos'd to flow uniformly, the Fluxions of x and x^n, will be as 1 and nx^{n-1}. For in the same Time that x by flowing, becomes $x+o$, x^n will become $\overline{x+o}|^n$, which by the Method of infinite Series, is equal to

$$x^n + nox^{n-1} + \frac{n^2-n}{2}\,o^2x^{n-2} + \textit{\&c.}$$

Consequently the Increments of x and x^n, generated in the same Time, are o and

$$nox^{n-1} + \frac{n^2-n}{2}\,o^2x^{n-2} + \textit{\&c.}$$

But the nascent or evanescent Increment of x^n, is as its Fluxion; and in either of these States the Ratio of

$$nox^{n-1} + \frac{n^2-n}{2}\,o^2x^{n-2} + \textit{\&c.}$$

to nox^{n-1} is a Ratio of Equality: For as the Magnitude of o becomes less and less, the Quantities

$$nox^{n-1} + \frac{n^2-n}{2}\,o^2x^{n-2} + \&c.$$

and nox^{n-1} constantly tend to an Equality, by a continual Diminution of their Difference; and they become equal, and their Ratio becomes a Ratio of Equality, when their Difference vanishes; that is, in the Instant o becomes evanescent, or in the Instant that the Increment of x^n first begins to exist: For as they vanish together under a Ratio of Equality, so they begin to exist together under the same Ratio; and therefore in the nascent or evanescent State of o, the Fluxions of x and x^n, are as o and nox^{n-1}, which are manifestly to each other as 1 to nx^{n-1}. Hence it appears, that this Method of finding the Fluxion of x^n, upon a Supposition that x flows uniformly, is the very same with that of finding the Fluxion of a Rectangle, as it is described in the second Book of the mathematical Principles: For, as $a\,b$ the Difference between $\mathrm{A}\,b + \mathrm{B}\,a + a\,b$ and $\mathrm{A}\,b + \mathrm{B}\,a$ grows less and less perpetually, by diminishing the synchronal Increments of the Sides of the Rectangle, and at last evanesces, and in the Instant of its Evanescence, the Gnomon $\mathrm{A}\,b + \mathrm{B}\,a + a\,b$ becomes equal to the Sum of the Rectangles $\mathrm{A}\,b$ and $\mathrm{B}\,a$; so $\frac{n^2-n}{2}\,o^2x^{n-2} + \&c.$, the Difference between

$$nox^{n-1} + \frac{n^2-n}{2}\,o^2x^{n-2} + \&c.$$

and nox^{n-1} grows less and less perpetually, by diminishing the Increment o, and at last evanesces, and in the Instant of its Evanescence

$$nox^{n-1} + \frac{n^2-n}{2}\,o^2x^{n-2} + \&c.$$

becomes equal to nox^{n-1}: And as the Gnomon $\mathrm{A}\,b + \mathrm{B}\,a + a\,b$ is not the Moment or Fluxion of the Rectangle $\mathrm{A\,B}$, but in the Instant of its becoming equal to $\mathrm{A}\,b + \mathrm{B}\,a$, so

$$nox^{n-1} + \frac{n^2-n}{2}\,o^2x^{n-2} + \&c.$$

is not the Moment or Fluxion of x^n, but in the Instant of its becoming equal to nox^{n-1},

The Author of the *Analyst* therefore, is greatly mistaken, in thinking Sir *Isaac* found the Fluxion of x^n, by a Method different from that he used in finding the Fluxion of a Rectangle, contain'd under two flowing Quantities: He steadily adheres to one and the same Method; namely, that of taking the first or last Ratios of Quantities, or of their isochronal Increments, for the Measures of the Ratios of their Fluxions; and uses no illegitimate Artifice to obtain these first or last Ratios; unless it be accounted illegitimate to suppose that mathematical Quantities can be generated and destroyed by Motion.

It is pretended, "That the Method for finding the Fluxion of a Rectangle of two flowing Quantities, as it is set forth in the *Treatise of Quadratures*, differs from that found in the *second Book of the Principles*, and is in Effect the same with that used in the *Calculus differentialis*: For the supposing a Quantity infinitely diminish'd and therefore rejecting it, is in Effect the rejecting an Infinitesimal." But if this Author deduces the Rule from the first Proposition in the *Treatise of Quadratures*, and considers it ever so little, he will find it the very same with that set down in the *second Book of the Principles*: And it is doubtless in Effect too the same with that used in the *differential Calculus*, so far as different Methods *can* effect the same Thing, but no farther: For Quantities are not rejected in the Method of Fluxions, as in the *differential Calculus*, on Account of their exceeding Smallness.

"But according to the received Principles it is evident," says he, "that no geometrical Quantity, by being infinitely diminish'd can ever be exhausted or become nothing." Now, on the received Principles of Fluxions, that is a direct Absurdity. For the Principles suppose that mathematical Quantities can be generated by Motion, which he has not yet thought proper to contradict; and consequently they may also by Motion be destroy'd: For Quantities, and the Augments of Quantities, which in some finite Time are produc'd by Motion, may perpetually grow less and less by reverting that Motion; and by constantly growing less and less, they may come to be infinitely diminished, or to be less than any assignable Quantities; and from being less than any assignable Quantities, the Motion still persevering, they must at last vanish and become nothing; otherwise it might be contended that a Body setting out from any Place, and, in any finite Time, describing a certain length, cou'd never by moving back and returning in the same Line, arrive at the Place from whence it first set out.

Upon the whole then it appears, that the Method of Fluxions, as describ'd by Sir *Isaac Newton*, in his Introduction to the *Quadrature of Curves*, and in the *second Book of his mathematical Principles*, is not that wretched un-scientifical Knack set forth in the *Analyst*; but a Method founded upon obvious, rational, accurate and demonstrative Principles: It likewise appears, that the Conclusions do not arise from illegitimate tentative Ways or Inductions, but follow from such Premises, and by such Arguments, as are most conformable to the Rules of Logic and right Reason: All the Skill and Dexterity therefore by this Author shewn, in the Investigation of contrary Errors correcting each other, are vain and impertinent. He has mistaken the Doctrine of Fluxions, and by not rightly distinguishing its Principles from those of the *differential Calculus*, has imposed a false Measure of Moments upon his Readers, and arguing from that false Measure, has unjustly charg'd Sir *Isaac* with Errors arising from it; and, to mend the Matter, has instituted Computations to shew how those Errors redress one another, and how Mathematicians by Means of Errors bring forth Truth and Science.

The Dispute between the Followers of Sir *Isaac Newton*, and the Author of the *Analyst*, is not about the Principles of the *differential Calculus*, but about those of Fluxions; and it is whether these Principles in themselves are clear or obscure, and whether the Inferences from them are just or unjust, true or false, scientific or otherwise: We are not concerned about Infinitesimals or minute Differences, but about the Ratios with which mathematical Quantities begin or cease to exist by Motion; and to consider the first or last Proportions of Quantities does not imply that such Quantities have any finite Magnitudes: They are not the Proportions of first or last Quantities, but Limits of Ratios; which Limits, the Ratios of Quantities attain *only* by an infinite Diminution of their Magnitudes, by which infinite Diminution of their Magnitudes they become evanescent and cease to exist. If therefore Quantities may cease to exist by Motion, and if the Ratios with which they become evanescent be truly determin'd, it will follow that there are no Errors, however small, admitted in the Principles of Fluxions; and if no Errors be admitted in the Principles; there can be none in the Conclusions, not any to be accounted for in the Arguments by which those Conclusions are deduc'd from their Premises: The Hints therefore, which this Author has condescended to give the Mathematicians for ascertaining the Truth of their Conclusions, by means of contrary Errors destroying each other, will probably be left to be further extended and apply'd by himself, to all the *good Purposes* he pleases to extend and apply them; as having more Leisure, and a Science more transcendental*, and perhaps a much greater Curiosity for such Matters, than they have.

It has been observ'd before, that Fluxions may be expounded by any Lines which are proportional to them; and so the Analysis may be instituted, by considering the mutual Relations or Proportions of finite Quantities, as the Proportions of Fluxions themselves. To this it is objected, "That if, in order to arrive at these finite Lines proportional to Fluxions, there be certain Steps made use of which are obscure and inconceivable, it must be acknowleged, that the Proceeding is not clear, nor the Method scientific." But there may be many Steps obscure and inconceivable to Persons, who are unacquainted with Sir *Isaac Newton*'s Method of first and last Ratios, with his Doctrine of Fluxions, and with his Principles of Motion; and yet those Steps may appear very different to others who have duly consider'd 'em: And therefore, till it be made apparent from geometrical Principles that the fluxional Triangle, which evanesces upon the returning of the Ordinate of any Curve to the Place from whence it first set out, cannot in its last Form, that is, in the Form it has at the Instant it becomes evanescent, be similar to a Triangle contain'd between the Tangent, the Absciss extended and the Ordinate of the same Curve; or till it be proved that no Triangle, which is capable of becoming evanescent by a Diminution of its Sides from Motion, can be similar in its last Form to any plain Triangle whatsoever; we shall continue to expound Fluxions by such Right Lines as are proportional to them; and do assert, that the Proceeding is clear, and the Method scientific.

FINIS.

* A *Philosophia prima*, a certain transcendental Science superior to and more extensive than Mathematics, which, he says, it might behove our modern *Analysts* rather to learn than despise.