A DISCOURSE CONCERNING THE NATURE AND CERTAINTY OF SIR ISAAC NEWTON'S METHODS OF FLUXIONS, AND OF PRIME AND ULTIMATE RATIOS

By

Benjamin Robins

Edited by David R. Wilkins

2002

NOTE ON THE TEXT

A Discourse Concerning the Nature and Certainty of Sir Isaac Newton's Methods of Fluxions, and of Prime and Ultimate Ratios, by Benjamin Robins was first published by W. Innys and R. Manby (London, 1735). This original 1735 edition has been used as the copy text for the present edition.

This *Discourse* was subsequently included in *Mathematical Tracts of the late Benjamin Robins Esq.*, edited by James Wilson, M.D., and published in London in 1761. There are certain small changes of wording and additional footnotes in this later posthumous edition; these have not been incorporated in the present edition.

The paragraphs in the original 1735 edition were unnumbered, and page references were used. This edition adopts the paragraph numbering employed in *Mathematical Tracts of the late Benjamin Robins Esq*, and replaces page references in the table of contents, in the body of the text, and in footnotes, with corresponding paragraph references.

The present text corrects the errata (in [21.], [23.] and [52.]) that were noted at the end of the original 1735 edition.

Two errata in [124.] have also been corrected; these were given by Robins in the October 1735 issue of *The Present State of the Republick of Letters*. The first sentence of this paragraph originally read as follows:

> If there be two quantities, that are (one or both) continually varying, either by being continually augmented, or continually diminished; and if the proportion, they bear to each other, does by this means perpetually vary,...

In addition, some obvious errors in the original text have been corrected in this edition, many of which had also been corrected in *Mathematical Tracts of the late Benjamin Robins Esq*. These corrections are noted below.

In [10.], [60.] and [155.], page references have been replaced by paragraph references, as in the version reprinted in *Mathematical Tracts of the late Benjamin Robins Esq*.

In [17.], superfluous commas have been removed before the occurrences of x in the numerators of some of the fractions. (That such commas are superfluous is confirmed by the fact that they are absent on subsequent recurrences of these formulæ in the original text.)

In [25.], an occurrence of $n - 1$ has been corrected to $\overline{n-1}$. (This correction was made in *Mathematical Tracts of the late Benjamin Robins Esq.*)

In [25.] and [38.], the current usual form '$-$' of the minus sign is used throughout, where the original 1735 edition employs (though not consistently) both '$-$' and a variant form, which takes the form of the letter 'S' turned on its side.

In [60.] 'x_n' has been corrected to read 'x^n'. (This correction was made in *Mathematical Tracts of the late Benjamin Robins Esq.*)

In [139.], the subformula '$3\,\overline{\mathrm{A\,F} + \mathrm{F\,G}}$' has been corrected to read '$\overline{3\,\mathrm{A\,F} + \mathrm{F\,G}}$'. (This correction was made in *Mathematical Tracts of the late Benjamin Robins Esq.*)

In [141.], the subformula '$3\,\overline{\mathrm{C\,F}+\mathrm{F\,I}}$' has been corrected to read '$\overline{3\,\mathrm{C\,F}+\mathrm{F\,I}}$'. (This correction was made in *Mathematical Tracts of the late Benjamin Robins Esq.*)

In [149.], the equation '$x_3 - xy^2 + a_2 z - b^3 = 0$' has been corrected so as to read '$x^3 - xy^2 + a^2 z - b^3 = 0$'. (This correction was made in *Mathematical Tracts of the late Benjamin Robins Esq.*)

The following spellings, differing from modern British English, are employed in the original 1735 edition: streight, preceeding, compleat, center, Euclide, inabled, surprizing, remembred.

The treatise referred to in [1.] was identified by James Wilson in *Mathematical Tracts of the late Benjamin Robins Esq.*, in a footnote, as 'Apollon. de Sectione Rationis, published by Dr. Halley at Oxford in 1706'.

Robins uses the notations $\mathrm{A\,B}\,q$ $\mathrm{A\,B}\,c$, $\mathrm{A\,B}\,qq$ to denote the square, cube, and fourth power respectively of a line segment such as A B. Analogous notation is adopted in Isaac Newton's *Philosophiæ Naturalis Principia Mathematica*.

Robins also employs the standard eighteenth century algebraic notation in which overlines are used for grouping terms within formulæ (where parentheses would today be employed).

David R. Wilkins
Dublin, June 2002

A DISCOURSE Concerning the NATURE and CERTAINTY OF Sir Isaac Newton's METHODS OF FLUXIONS, AND OF Prime and Ultimate RATIOS.

BY

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THE

CONTENTS.

INTRODUCTION.

[1.] From many propositions dispersed through the writings of the ancient geometers, and more especially from one whole treatise, it appears, that the process, by which they investigated the solutions of their problems, was for the most part the reverse of the method, whereby they demonstrated those solutions. But what they have delivered upon the tangents of curve lines, and the mensuration of curvilinear spaces, does not fall under this observation; for the analysis, they made use of in these cases, is no where to be met with in their works. In later times, indeed, a method for investigating such kind of problems has been introduced, by considering all curves, as composed of an infinite number of indivisible streight lines, and curvilinear spaces, as composed in the like manner of parallelograms. But this being an obscure and indistinct conception, it was obnoxious to error.

[2.] Sir Isac Newton therefore, to avoid the imperfection, with which this method of indivisibles was justly charg'd, instituted an analysis for these problems upon other principles. Considering magnitudes not under the notion of being increased by a repeated accession of parts, but as generated by a continued motion or flux; he discovered a method to compare together the velocities, wherewith homogeneous magnitudes increase, and thereby has taught an analysis free from all obscurity and indistinctness.

[3.] Moreover to facilitate the demonstrations for these kinds of problems, he invented a synthetic form of reasoning from the prime and ultimate ratios of the contemporaneous augments, or decrements of those magnitudes, which is much more concise than the method of demonstrating used in these cases by the ancients, yet is equally distinct and conclusive.

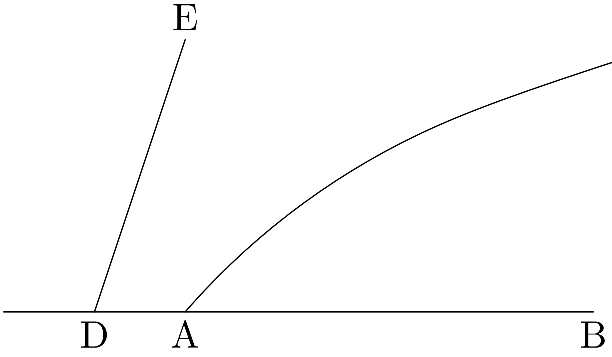
[4.] Of this analysis, called by Sir Isaac Newton his method of fluxions, and of his doctrine of prime and ultimate ratios, I intend to write in the ensuing discourse. For though Sir Isaac Newton has very distinctly explained both these subjects, the first in his treatise on the Quadrature of curves, and the other in his Mathematical principles of natural philosophy; yet as the author's great brevity has made a more diffusive illustration not altogether unnecessary; I have here endeavoured to consider more at large each of these methods; whereby, I hope, it will appear, they have all the accuracy of the strictest mathematical demonstration.

OF
FLUXIONS.

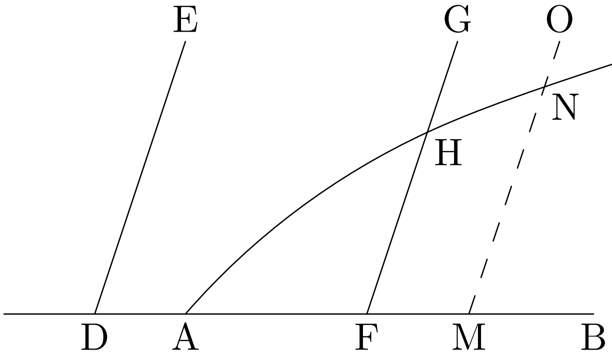
[5.] In the method of fluxions geometrical magnitudes are not presented to the mind, as compleatly formed at once, but as rising gradually before the imagination by the motion of some of their extremes*.

[6.] Thus the line A B may be conceived to be traced out gradually by a point moving on

from A to B, either with an equable motion, or with a velocity in any manner varied. And the velocity, or degree of swiftness, with which this point moves in any part of the line A B, is called the fluxion of this line at that place.



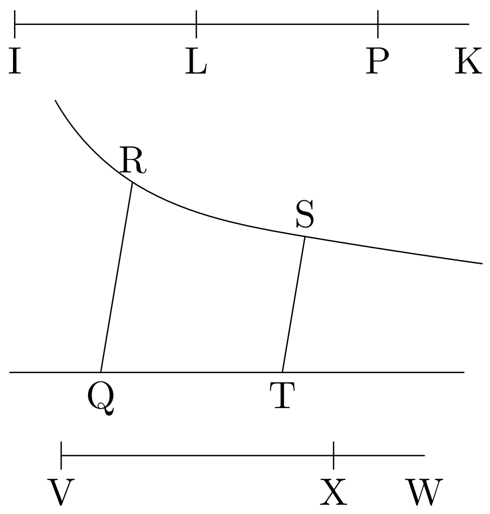
[7.] Again, suppose two lines A B and A C to form a space unbounded towards B C; and upon A B a line D E to be erected.



* Newt. Introd. ad Quad. Curv.

[8.] NOW, if this line D E be put in motion (suppose so as to keep always parallel to itself,) as soon as it has passed the point A, a space bounded on all sides will begin to appear between these three lines. For instance, when D E is moved into the situation F G, these three lines will include the space A F H. Here it is evident, that this space will increase faster or slower, according to the degree of velocity, wherewith the line D E shall move. It is also evident, that though the line D E should move with an even pace, the space A F H would not for that reason increase equably; but where the line A C was farthest distant from A B, the space A F H would increase fastest. Now the velocity or celerity, wherewith the space A F H at all times increases, is called the fluxion of that space.

[9.] HERE it is obvious, that the velocity, wherewith the space augments, is not to be understood literally the degree of swiftness, with which either the line F G, or any other line or point appertaining to the curve actually moves; but as this space, while the line F G moves on uniformly, will increase more, in the same portion of time, at some places, than at others; the terms velocity and celerity are applied in a figurative sense to denote the degree, wherewith this augmentation in every part proceeds.



[10.] BUT we may divest the consideration of the fluxion of the space from this figurative phrase, by causing a point so to pass over any streight line I K, that the length I L measured out, while the line D E is moving from A to F shall augment in the same proportion with the space A F H. For this line being thus described faster or slower in the same proportion, as the space receives its augmentation; the velocity or degree of swiftness, wherewith the point describing this line actually moves, will mark out the degree of celerity, wherewith the space every where increases. And here the line I L will preserve always the same analogy to the space A F H, in so much, that, when the line D E is advanced into any other situation M N O, if I P be to I L in the proportion of the space A M N to the space A F H, the fluxion of the space at M N will be to the fluxion thereof at F H, as the velocity, wherewith the point describing the line I K moves at P, to the velocity of the same at L. And if any other space Q R S T be described along with the former by the like motion, and at the same time a line V W, so that the portion V X shall always have to the length I L the same proportion, as the space Q R S T bears to the space A F H; the fluxion of this latter space at T S will be to the fluxion of the former at F H, as the velocity, wherewith the line V W is described at X, to the velocity, wherewith the line I K is described at L. It will hereafter appear, that in all the applications of fluxions to geometrical problems, where spaces are concerned, nothing more is necessary, than to determine the velocity, wherewith such lines as these are described*.

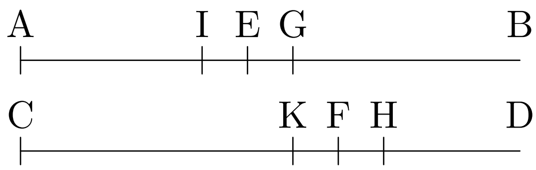
[11.] In the same manner may a solid space be conceived to augment with a continual flux, by the motion of some plane, whereby it is bounded; and the velocity of its augmentation (which may be estimated in like manner) will be the fluxion of that solid.

[12.] Fluxions then in general are the velocities, with which magnitudes varying by a continued motion increase or diminish; and the magnitudes themselves are reciprocally called the fluents of those fluxions†.

[13.] And as different fluents may be understood to be described together in such manner, as constantly to preserve some one known relation to each other; the doctrine of fluxions teaches, how to assign at all times the proportion between the velocities, wherewith homogeneous magnitudes, varying thus together, augment or diminish.

[14.] This doctrine also teaches on the other hand, how from the relation known between the fluxions, to discover what relation the fluents themselves bear to each other.

[15.] It is by means of this proportion only, that fluxions are applied to geometrical uses; for this doctrine never requires any determinate degree of velocity to be assigned for the fluxion of any one fluent. And that the proportion between the fluxions of magnitudes is assignable from the relation known between the magnitudes themelves, I now proceed to shew.



[16.] In the first place, let us suppose two lines A B and C D to be described together by two points, one setting out from A, and the other from C, and to move in such manner, that if A E and C F are lengths described in the same time, C F shall be analogous to some power of A E, that is, if A E be denoted by the letter x, then C F shall always be denoted by $\dfrac{x^n}{a^{n-1}}$; where a represents some given line, and n any number whatever. Here, I say, the proportion between the velocity of the point moving on A B to the velocity of that moving on C D, is at all times assignable.

[17.] For let any other situations, that these moving points shall have at the same instant of time, be taken, either farther advanced from E and F, as at G and H, or short of the same, as at I and K; then if E G be denoted by e, C H, the length passed over by the

* [§ 49.]

† Motuum vel incrementorum velocitates nominando fluxiones, & quantitates genitas nominando fluentes. Newton. Introd. ad Quadr. Curv.

point moving on the line C D, while the point in the line A B has passed from A to G, will be expressed by $\frac{\overline{x+e}|^n}{a^{n-1}}$; and if E I be denoted by e, C K, the length passed over by the point moving on the line C D, while the point moving in A B has got only to I, will be denoted by $\frac{\overline{x-e}|^n}{a^{n-1}}$: or reducing each of these terms into a series, C H will be denoted by

$$\frac{x^n}{a^{n-1}} + \frac{nx^{n-1}e}{a^{n-1}} + \frac{n \times \overline{n-1}\,x^{n-2}ee}{2a^{n-1}} + \frac{n \times \overline{n-1} \times \overline{n-2}\,x^{n-3}e^3}{6a^{n-1}} + \&\text{c.}$$

and C K by

$$\frac{x^n}{a^{n-1}} - \frac{nx^{n-1}e}{a^{n-1}} + \frac{n \times \overline{n-1}\,x^{n-2}ee}{2a^{n-1}} - \frac{n \times \overline{n-1} \times \overline{n-2}\,x^{n-3}e^3}{6a^{n-1}} + \&\text{c.}$$

Hence all the terms of the former series, except the first term, *viz.*

$$\frac{nx^{n-1}e}{a^{n-1}} + \frac{n \times \overline{n-1}\,x^{n-2}ee}{2a^{n-1}} + \&\text{c.}$$

will denote F H; and all the latter series, except the first term, *viz.*

$$\frac{nx^{n-1}e}{a^{n-1}} - \frac{n \times \overline{n-1}\,x^{n-2}ee}{2a^{n-1}} + \&\text{c.}$$

will denote K F.

[18.] WHEN the number n is greater than unite, while the line A B is described with a uniform motion, the point, wherewith C D is described, moves with a velocity continually accelerated, for if I E be equal to E G, F H will be greater than K F.

[19.] NOW here, I say, that neither the proportion of F H to E G, nor the proportion of K F to I E is the proportion of the velocity, which the point moving on C D has at F, to the uniform velocity, wherewith the point moves on the line A B. For, while that point is advanced from E to G, the point moving on C D has passed from F to H, and has moved through that space with a velocity continually accelerated; therefore, if it had moved during the same interval of time with the velocity, it has at F, uniformly continued, it would not have passed over so long a line; consequently F H bears a greater proportion to E G, than what the velocity, which the point moving on C D has at F, bears to the velocity of the point moving uniformly on A B.

[20.] IN like manner K F bears to I E a less proportion than that, which the velocity of the point in C D has at F, to the velocity of that in A B. For as the point in C D, in moving from K to F, proceeds with a velocity continually accelerated; with the velocity, it has acquired at F, if uniformly continued, it would describe in the same space of time a line longer than K F.

[21.] IN the last place I say, that no line whatever, that shall be greater or less than the line represented by the second term of the foregoing series (*viz.* the term $\frac{nx^{n-1}e}{a^{n-1}}$) will bear to the line denoted by e the same proportion, as the velocity, wherewith the point moves at F, bears to the velocity of the point moving in the line A B; but that the velocity at F is to that at E as $\frac{nx^{n-1}e}{a^{n-1}}$ to e, or as nx^{n-1} to a^{n-1}.

[22.] If possible let the velocity at F bear to the velocity at E a greater ratio than this, suppose the ratio of p to q.

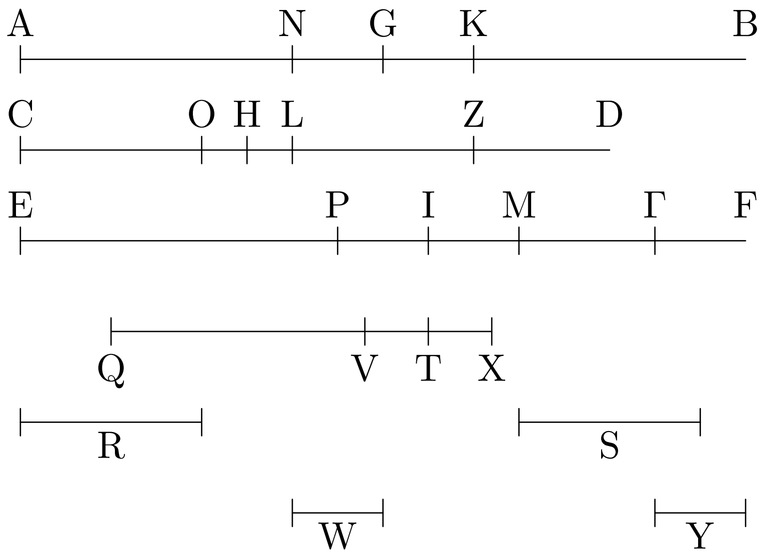
[23.] In the series, whereby CH is denoted, the line e can be taken so small, that any term proposed in the series shall exceed all the following terms together; so that the double of that term shall be greater than the whole collection of that term, and all that follow. Again, by diminishing e, the ratio of the second term in this series to twice the third, that is, of $\frac{nx^{n-1}e}{a^{n-1}}$ to $\frac{n\times\overline{n-1}\,x^{n-2}ee}{a^{n-1}}$ or the ratio of x to $\overline{n-1}\times e$, shall be greater than any, that shall be proposed; consequently the line e may be taken so small, that twice the third term, that is $\frac{n\times\overline{n-1}\,x^{n-2}ee}{a^{n-1}}$, shall be greater than all the terms following the second, and also, that the ratio of $\frac{nx^{n-1}e}{a^{n-1}}+\frac{n\times\overline{n-1}\,x^{n-2}ee}{a^{n-1}}$ to e shall less exceed the ratio of $\frac{nx^{n-1}e}{a^{n-1}}$ to e, than any other ratio, that can be proposed. Therefore let the ratio of $\frac{nx^{n-1}e}{a^{n-1}}+\frac{n\times\overline{n-1}\,x^{n-2}ee}{a^{n-1}}$ to e be less than the ratio of p to q; then, if $\frac{n\times\overline{n-1}\,x^{n-2}ee}{a^{n-1}}$ be also greater than the third and all the following terms of the series, the ratio of the series $\frac{nx^{n-1}e}{a^{n-1}}+\frac{n\times\overline{n-1}\,x^{n-2}ee}{2a^{n-1}}+$ &c. to e, that is, the ratio of FH to EG shall be less than the ratio of p to q, or of the velocity at F to the velocity at E, which is absurd; for it has above been shewn, that the first of these ratios is greater than the last. Therefore the velocity at F cannot bear to the velocity at E any greater proportion than that of $\frac{nx^{n-1}e}{a^{n-1}}$ to e.

[24.] On the other hand, if possible, let the velocity at F bear to the velocity at E a less ratio than that of $\frac{nx^{n-1}e}{a^{n-1}}$ to e: let this latter ratio be that of r to s.

[25.] In the series whereby CK is denoted, e may be taken so small, that any one term proposed shall exceed the whole sum of all the following terms, when added together. Therefore let e be taken so small, that the third term $\frac{n\times\overline{n-1}\,x^{n-2}ee}{2a^{n-1}}$ exceed all the following terms $\frac{n\times\overline{n-1}\times\overline{n-2}\,x^{n-3}e^3}{6a^{n-1}}$, $\frac{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\,x^{n-4}e^4}{24a^{n-1}}$, &c. added together. But e may also be so small, that the ratio of $\frac{nx^{n-1}e}{a^{n-1}}$ to $\frac{n\times\overline{n-1}\,x^{n-2}ee}{a^{n-1}}$, the double of the third term, shall be greater than any ratio, that can be proposed; and the ratio of $\frac{nx^{n-1}e}{a^{n-1}}-\frac{n\times\overline{n-1}\,x^{n-2}ee}{a^{n-1}}$ to e shall come less short of the ratio of $\frac{nx^{n-1}e}{a^{n-1}}$ to e, than any other ratio, that can be named. Therefore let this ratio exceed the ratios of r to s; then the term $\frac{n\times\overline{n-1}\,x^{n-2}ee}{2a^{n-1}}$ exceeding the whole sum of all the following terms in the series denoting CK, the whole series $\frac{nx^{n-1}e}{a^{n-1}}-\frac{n\times\overline{n-1}\,x^{n-2}ee}{2a^{n-1}}+$ &c. or KF, will in every case bear to e, or EI a greater ratio than that of r to s, or of the velocity at F to the velocity at E, which is absurd. For it has above been shewn, that the first of these ratios is less than the last.

[26.] If n be less than unite, the point in the line C D moves with a velocity continually decreasing; and if n be a negative number, this point moves backwards. But in all these cases the demonstration proceeds in like manner.

[27.] Thus have we here made appear, that from the relation between the lines A E and C F, the proportion between the velocities, wherewith they are described, is discoverable; for we have shewn, that the proportion of nx^{n-1} to a^{n-1} is the true proportion of the velocity, wherewith C F, or $\dfrac{x^n}{a^{n-1}}$ augments, to the velocity, wherewith A E, or x is at the same time augmented.



[28.] Again, in the three lines A B, C D, E F, where the points A, C, E are given, let us suppose G, H and I to be three contemporary positions of the points, whereby the three lines A B, C D, E F are respectively described; and let the motion of the point describing the line E F be so regulated with regard to the motion of the other two points, that the rectangle under E I and some given line may be always equal to the rectangle under A G and C H. Here from the velocities, or degrees of swiftness, wherewith the points describing A B and C D move, the degree of swiftness, wherewith the point describing E F moves, may be determined.

[29.] The points moving on the lines A B, C D may either move both the same way, or one forwards and the other backwards.

[30.] In the first place suppose them to move the same way, advancing forward from A and C; and since some given line forms with E I a rectangle equal to that under A G and C H, suppose $\mathrm{QT} \times \mathrm{EI} = \mathrm{AG} \times \mathrm{CH}$; then, if K, L, M are contemporary positions of the points moving on the lines A B, C D, E F, when advanced forward beyond G, H and I; and N, O, P, three other contemporary positions of the same points, before they are arrived at G, H and I; $\mathrm{QT} \times \mathrm{EM}$ will also be $= \mathrm{AK} \times \mathrm{CL}$, and $\mathrm{QT} \times \mathrm{EP} = \mathrm{AN} \times \mathrm{CO}$; therefore the rectangle under I M (the difference of the lines E I and E M) and Q T will be $= \mathrm{AK} \times \mathrm{HL} + \mathrm{CH} \times \mathrm{GK}$, and $\mathrm{IP} \times \mathrm{QT} = \mathrm{AN} \times \mathrm{HO} + \mathrm{CH} \times \mathrm{GN}$.

[31.] HERE the proportion of the velocity, which the point moving on AB has at G, to that, which the point moving on CD has at H, may either keep always the same, or continually vary, and one of these velocities, suppose that of the point moving on the line CD, have to the other a proportion gradually augmenting; that is, if NG and GK are equal, HL shall either be equal to OH or greater. Here, since $IM \times QT$ is $= AK \times HL + CH \times GK$, and $IP \times QT = AN \times HO + CH \times GN$, where $CH \times GK$ is $= CH \times GN$ and $AK \times HL$ in both cases greater than $AN \times HO$, IM will be greater than IP; in so much that in both these cases the velocity of the point, wherewith the line EF is described, will have to the velocity of the point moving on AB a proportion, gradually augmenting. Here therefore the line IM will bear to GK a greater proportion, than the velocity of the point moving on the line EF, when at I, bears to the velocity of the point moving on the line AB, when at G: and the line PI will have a less proportion to NG, than the velocity, which the point moving on the line EF, has at I, to the velocity, which the point moving on the line AB has at G.

[32.] NOW let R be to S as the velocity, which the point moving on AB has at G, to the velocity, which the point moving on CD has at H; then I say, that the velocity, which the point moving on EF has at I, will be to the velocity, which the point moving on AB has at G, as $AG \times S + CH \times R$ to $QT \times R$.

[33.] IF possible let the velocity, which the point moving on EF has at I, be to the velocity, which the point moving on AB has at G, as $AG \times S + CH \times R$ to the rectangle under R and some line QV less than QT.

[34.] TAKE W to GK in the ratio of S to R; then will $AG \times S + CH \times R$ be to $R \times QV$ as $AG \times W + CH \times GK$ to $QV \times GK$. Here, because the ratio of the velocity of the point moving on the line CD to the velocity of the point moving on AB either remains constantly the same, or gradually augments, W is either equal to HL or less; but when it is less, by diminishing HL the ratio of W to HL may become greater than any ratio, that can be proposed, short of the ratio of equality. The like is true of the ratio of AG to AK by the diminution of GK. Therefore let GK and HL be so diminished, that the ratio of $AG \times W$ to $AK \times HL$ shall be greater than the ratio of QV to QT; then the ratio of $AG \times W + CH \times GK$ to $AK \times HL + CH \times GK$, that is, to $QT \times IM$ is greater than the ratio of QV to QT or of $QV \times IM$ to $QT \times IM$; therefore $AG \times W + CH \times GK$ is greater than $QV \times IM$; and the ratio of $AG \times W + CH \times GK$ to $QV \times GK$ is greater than the ratio of $QV \times IM$ to $QV \times GK$, or of IM to GK; but the ratio of IM to GK is greater than that of the velocity, which the point moving on EF has at I, to the velocity, which the point moving on AB has at G; therefore the ratio of $AG \times W + CH \times GK$ to $QV \times GK$, or that of $AG \times S + CH \times R$ to $QV \times R$, still more exceeds the ratio of the velocity at I to the velocity at G; and consequently the ratio of the velocity at I to the velocity at G is not greater than that of $AG \times S + CH \times R$ to $QT \times R$.

[35.] AGAIN, if possible let the velocity, which the point moving on EF has at I, be to the velocity, which the point moving on AB has at G, as $AG \times S + CH \times R$ to the rectangle under R and some line QX greater than QT.

[36.] HERE let Y be to NG as S to R; then will $AG \times S + CH \times R$ be to $R \times QX$ as $AG \times Y + CH \times NG$ to $QX \times NG$. But Y will be either greater than HO, or equal to it, and when greater, by diminishing H O, the ratio of Y to H O may become less than any ratio, that can be proposed, greater than the ratio of equality. The like is true of the ratio of A G to A N by the diminution of N G. Therefore let N G and H O be so diminished, that the ratio of $AG \times Y$ to $AN \times HO$ shall be less than the ratio of Q X to Q T; then the ratio of $AG \times Y + CH \times NG$ to $AN \times HO + CH \times NG$, that is, to $QT \times IP$, is less than the ratio of Q X to Q T, or of $QX \times IP$ to $QT \times IP$. Consequently $AG \times Y + CH \times NG$ is less than $QX \times IP$, and the ratio of $AG \times Y + CH \times NG$ to $QX \times NG$ is less than the ratio of $QX \times IP$ to $QX \times NG$, or of I P to N G. But the ratio of I P to N G is less than that of the velocity, which the point moving on E F has at I, to the velocity, which the point moving on A B has at G. Therefore the ratio of $AG \times Y + CH \times NG$ to $QX \times NG$, or that of $AG \times S + CH \times R$ to $QX \times R$, is also less than the ratio of the velocity at I to the velocity at G. Consequently, the ratio of the velocity at I to the velocity at G is not less than that of $AG \times S + CH \times R$ to $QT \times R$.

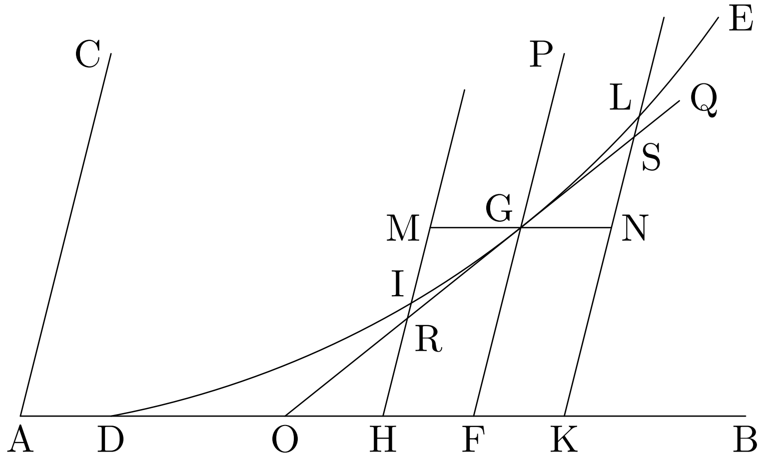
[37.] If the points describing A B and C D move backwards together, the velocity at I will be the same, and the demonstration will proceed in like manner.

[38.] But if one of these points, as that moving on C D, recedes, while the other on A B advances forward, take in C D any fix'd point at pleasure Z; then the point on C D in respect of Z moves also forward. Again, take in the line E F, E Γ to A G as C Z to Q T; then $AG \times CZ$ is $= QT \times E\Gamma$; and $AG \times CH$ being $= QT \times EI$, $AG \times HZ$ will be $= QT \times \Gamma I$; and by the preceeding case $AG \times S + ZH \times R$ will be to $QT \times R$ as the velocity, wherewith the point moving on E G separates from Γ, when at I, to the velocity, which the point moving on A B has at G. But as A G is continually increasing, and E Γ keeps always in the same proportion to A G; the point Γ will itself be in motion, and the velocity of the point Γ will be to the velocity at G, as the line E Γ to A G, that is, as C Z to Q T, or as $CZ \times R$ to $QT \times R$; therefore the velocity, wherewith the point moving on E F, when at I, separates from Γ, being to the velocity of the point moving on A B, when at G, as $AG \times S + ZH \times R$ to $QT \times R$; the absolute velocity, which the point moving on E F has at I, will be to the absolute velocity, which the point moving on A B has at G, as $AG \times S - CH \times R$ to $QT \times R$; moving backwards, when it separates from Γ swifter than the point Γ itself moves, that is, when $AG \times S + ZH \times R$ is greater than $CZ \times R$, or $AG \times S$ greater than $CH \times R$; and when the point moving on E F, at I separates from Γ with a slower motion, than that wherewith Γ moves, that is, when $CZ \times R$ is greater than $AG \times S + ZH \times R$, or $AG \times S$ less than $CH \times R$, the point moving on E F, at I advances forward.

[39.] We have in our demonstrations only considered the fluxions of lines; but by these the fluxions of all other quantities are determined. For we have already observed, that the fluxions of spaces, whether superficial or solid, are analogous to the velocities, wherewith lines are described, that augment in the same proportion with such spaces.

[40.] Thus we have attempted to prove the truth of the rules, Sir Isaac Newton has laid down, for finding the fluxions of quantities, by demonstrating the two cases, on which all the rest depend, after a method, which from all antiquity has been allowed as genuine, and universally acknowledged to be free from the least shadow of uncertainty.

[41.] We shall hereafter endeavour to make manifest, that Sir Isaac Newton's own demonstrations are equally just with these, we have here exhibited. But first we shall prove, that in all the applications of this doctrine to the solution of geometrical problems, no other conception concerning fluxions is necessary, than what we have here given. And for this end it will be sufficient to shew, how fluxions are to be applied to the drawing of tangents to curve lines, and to the mensuration of curvilinear spaces.



[42.] If upon the line A B be erected in any angle another streight line A C, and it be put in motion upon the line A B towards B keeping always parallel to itself, and proceeding on with a uniform velocity: if a point also moves on the line A C with a velocity in any manner regulated; this point will describe within the angle under C A B some third line D E, which will be a curve, unless the point moves in the line A C likewise with a uniform motion.

[43.] Here, I say, the line A C being advanced to any situation F G, by what has already been written on the nature of fluxions, without any adventitious consideration whatever, a tangent may be assigned to the curve at the point G.

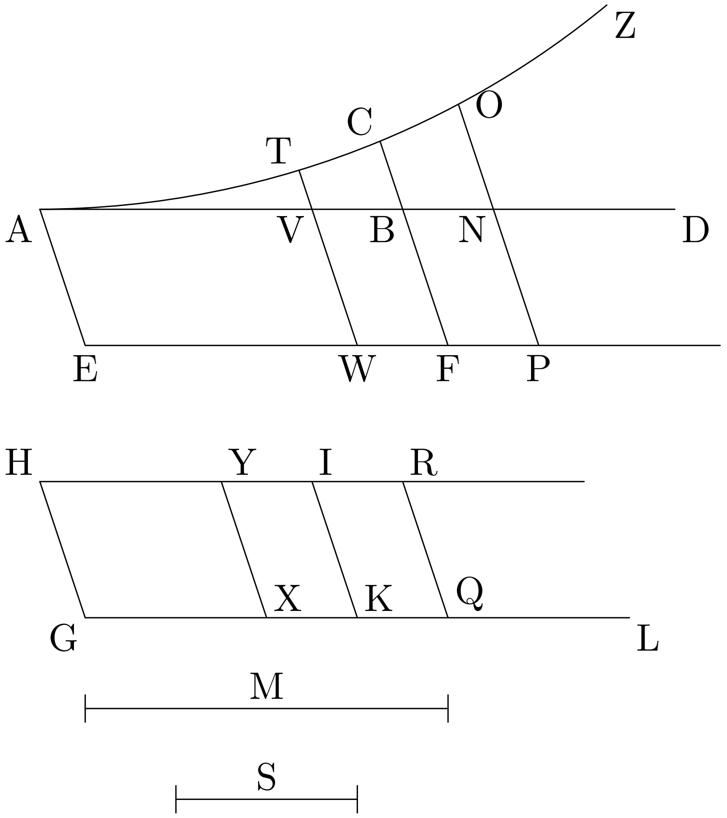
[44.] When the point moves on the line A C with an accelerating velocity, the curve D E will be convex to the abscisse D B. Now if two other situations H I and K L of the line A C be taken, one on each side F G, and M G N be drawn parallel to A B; while the line A C is moving from the situation H I to F G, the point in it will have moved through the length I M, and while the same line A C moves from F G to K L, the point in it will have passed over the length N L. And since the point moves with an accelerated velocity, I M will be less, and N L greater than the space, which would have been described in the same time by the velocity, the point has at G.

[45.] Let F O be taken to F G in the proportion of the velocity, wherewith the point F moves on the line A B, to the velocity, which the point moving on the line F P has at G, and the streight line O G Q be drawn, cutting H I in R, and K L in S; then F H will be to M R, and F K to N S in the same proportion. Therefore, from what has been said above, M R will be greater than M I, and N S less than N L; so that the line O Q, which unites with the curve at the point G, lies on both sides the point G, on the same side of the curve; that is, it does not cross, or cut the curve (as geometers speak) but touches it only at the point G.

[46.] When the point moves on the line A C with a velocity gradually decreasing, the curve will be concave towards the abscisse; but in this case the method of reasoning will be still the same.

[47.] IF the curve D E be the conical parabola, the latus rectum being T, and $\mathrm{T} \times \mathrm{F\,G} = \mathrm{D\,F}\,q$, or $\mathrm{F\,G} = \dfrac{\mathrm{D\,F}\,q}{\mathrm{T}}$; the fluxion of D F will be to the fluxion of $\dfrac{\mathrm{D\,F}\,q}{\mathrm{T}}$ (that is, the fluxion of F G) as T to 2 D F; therefore O F is to F G in the same proportion of T to 2 D F, or of D F to 2 F G, and O F is half D F.

[48.] IN like manner by the consideration of these velocities only may the mensuration of curvilinear spaces be effected.



[49.] SUPPOSE the curvilinear space A B C to be generated by the parallel motion of the line B C upon the line A D with a uniform velocity, within the space comprehended between the streight line A D and the curve line A Z; and let the parallelogram A E F B be generated with it by the motion of B F accompanying B C. Suppose another parallelogram G H I K to be generated at the same time by the motion of the line G H equal to A E or B F, insisting on the line G L in an angle equal to that under C B D; and let the motion of G H be so regulated, that the parallelogram G H I K be always equal to the curvilinear space A B C. Then it is evident, by what has been said above in our explanation of the nature of fluxions, that the velocity, wherewith the parallelogram E A B F increases, is to the velocity, wherewith the parallelogram G H I K, or wherewith the curvilinear space A B C increases; as the velocity, wherewith the point B moves, to the velocity, wherewith the point K moves.

[50.] NOW I say, the velocity of the point B is to the velocity of the point K as B F to B C.

[51.] Suppose the curve line A C Z to recede farther and farther from A D; then it is evident, that while the parallelogram E A B F augments uniformly, the curvilinear space A B C will increase faster and faster; therefore in this case the point K moves with a velocity continually accelerated.

[52.] Here, if possible, suppose the velocity of the point B to bear a less proportion to the velocity of the point K, than the ratio of B F to B C; that is, let the velocity of B be to the velocity of K, as B F to some line M greater than BC. Then it is possible to draw within the curve A C Z towards D a line, as O N, parallel to B C, which, though it exceed B C, shall be less than M; and the ratio of the velocity of the point B to the velocity of the point K, will be less than the ratio of B F to N O, or than the ratio of the parallelogram B P to the parallelogram B O; therefore still less than the ratio of the parallelogram B P to the space B C O N. Farther let the parallelogram K I R Q be taken equal to the space B C O N, then will the point K have moved from K to Q in the time, that the point B has moved from B to N. Now the parallelogram B P is to the parallelogram K R as B N to K Q, that is, as the velocity, wherewith the point B passes over B N, to the velocity, wherewith K Q would be described in the same time with a uniform motion. But as the point K moves with a velocity continually accelerated, its velocity at K is less than this uniform velocity now spoken of; therefore the velocity of the point B bears a greater proportion to the velocity of the point K than the parallelogram B P bears to the parallelogram K R; that is, than the parallelogram B P bears to the space B C O N; but the first of these ratios was before found less than the last; which involves an absurdity. Therefore the velocity of B bears not to the velocity of K a less proportion than that of B F to B C.

[53.] Again, if possible, let the velocity of B bear to the velocity of K a greater proportion than that of B F to B C, that is, the proportion of B F to some line S less than B C; and let the line T V be drawn parallel to C B, and greater than S, and the parallelogram T B be compleated. Here the ratio of the velocity of the point B to the velocity of the point K will be greater than the ratio of B F to T V, or than the ratio of the parallelogram B W to the parallelogram B T, therefore still greater than the ratio of the parallelogram B W to the curvilinear space V T C B. Now if the parallelogram X Y I K be taken equal to the space V T C B, that the point describing the line G L may have moved from X to K, while V T has moved to B C; since the parallelogram B W is to the parallelogram X I as V B to X K, that is, as the velocity, wherewith the point B has passed over V B, to the velocity, wherewith X K would be described in the same time with a uniform motion, the velocity of the point B bears a less proportion to the velocity of the point K, than the parallelogram B W bears to the parallelogram X I, because X K is described with an accelerating velocity: that is, the velocity of the point B bears a less proportion to the velocity of the point K, than the parallelogram B W bears to the space V T C B. But the first of these ratios was before found greater than the last. Therefore the velocity of B does not bear to the velocity of K a greater proportion than that of B F to B C.

[54.] If the curve line A C Z were of any other form, the demonstration would still proceed in the same manner.

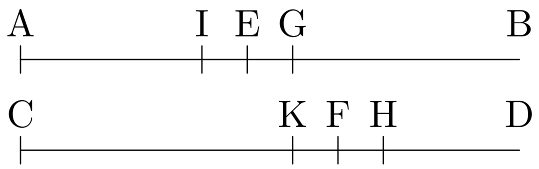
[55.] Hence it appears, that nothing more is necessary towards the mensuration of the curvilinear space A B C, than to find a line G K so related to A B, that, while they are described together, the velocity of the point, wherewith A B is described, shall bear the same proportion at any place B to the velocity, wherewith the point describing the other line G K moves at the correspondent place K, as some given line A E bears to the ordinate B C of the curve A C Z.

[56.] THE method of finding such lines is the subject of Sir Isaac Newton's Treatise upon the Quadrature of Curves.

[57.] FOR example, if A C Z be a conical parabola as before, and $\Gamma \times \mathrm{B\,C} = \mathrm{A\,B}\,q$; taking $\mathrm{G\,K} = \dfrac{\mathrm{A\,B}\,c}{3\,\Gamma \times \mathrm{G\,H}}$, the parallelogram $\mathrm{H\,K} = \dfrac{\mathrm{A\,B}\,c}{3\,\Gamma} = \frac{1}{3}\mathrm{A\,B} \times \mathrm{B\,C}$, is equal to the space A B C; for G K being equal to $\dfrac{\mathrm{A\,B}\,c}{3\,\Gamma \times \mathrm{G\,H}}$, the fluxion of G K or the velocity, wherewith it is described at K, will be to the fluxion of A B, or the velocity, wherewith B moves, as $\dfrac{\mathrm{A\,B}\,q}{\Gamma}$ or B C to G H or A E.

[58.] HAVING thus, as we conceive, sufficiently explained, what relates to the proportions between the velocities, wherewith magnitudes are generated; nothing now remains, before we proceed to the second part of our present design, but to consider the variations, to which these velocities are subject.

[59.] WHEN fluents are not augmented by a uniform velocity, it is convenient in many problems to consider how these velocities vary. This variation Sir Isaac Newton calls the fluxion of the fluxion, and also the second fluxion of the fluent; distinguishing the fluxions, we have hitherto treated of, by the name of first fluxions. These second fluxions may also vary in different magnitudes of the fluent, and the variation of these is called the third fluxion of the fluent. Fourth fluxions are the changes to which the third are subject, and so on*.



[60.] IN the two fluents A E and C F, whose fluxions we compared [at § 16, &c.] where A E being denoted by x, C F was equal to $\dfrac{x^n}{a^{n-1}}$, and the fluxion of A E bore to the fluxion of C F the proportion of a^{n-1} to nx^{n-1}. Here it is evident, that the antecedent a^{n-1} of this proportion being a fixed quantity, and the consequent nx^{n-1} a variable one; the fluxion of A E does not bear to the fluxion of C F always the same proportion. If n be the number 2, the fluxion of A E is to the fluxion of C F as a to the variable quantity $2x$; and if n be the number 3, the fluxion of A E to that of C F will be as a^2 to $3x^2$. Therefore if A E be described with an uniform velocity, when n is any number greater than unite, C F is so described with a velocity continually accelerating, that when $n = 2$, this velocity augments in the same proportion as C F itself increases; and when n is $= 3$, it augments in the duplicate of that proportion, &c.

* Fluxionum (scilicet primarum) fluxiones seu mutationes magis aut minus celeres fluxiones secundas nominare licet, &c. Newt. Quadr. Curv. in Princip.

[61.] HERE therefore we see, that while one quantity flows uniformly, the other is described with a varying motion; and the variation in this motion is called the second fluxion of this quantity.

[62.] IT is evident farther, that in this instance, when n is $= 2$, the variation of the velocity is uniform: for the velocity keeping always in the same proportion to x, while x increases uniformly, the velocity must also increase after the same manner. But when n is $= 3$; since the velocity is every where as x^2, and x^2 does not increase uniformly; neither will the velocity augment uniformly. So that it appears by this example, that the variation in the velocity, wherewith magnitudes increase, may also vary, and this variation is called the third fluxion of the magnitude.

[63.] IN the same manner may the fluxions of the folowing orders be conceived; each order being the variation found in the preceeding one. And the consideration of velocities thus perpetually varying, and their variation itself changing, is a useful speculation; for most, if not all, the bodies, we have any acquaintance with, do actually more with velocities thus modified.

[64.] A STONE, for instance, in its directed fall towards the earth has its velocity perpetually augmented; and in Galileo's Theory of falling Bodies, when the whole descent is performed near the surface of the earth, it is supposed to receive equal augmentations of velocity in equal times. In this case therefore the velocity augments uniformly, and the second fluxion of the line described by the falling body will in all parts of that line be the same; so that third fluxions cannot take place in this instance; since the variation of the velocity suffers no change, but is every where uniform.

[65.] BUT if the stone be supposed to have its gravity at the beginning of its fall less than at the surface of the earth, the variation of its velocity at first will then be less than the variation at the end of its motion; or in other words, the second fluxions in the beginning and end of its fall would be unequal; consequently, the third fluxions would here take place, since the variation would be swifter, as the body in its fall approached the earth.

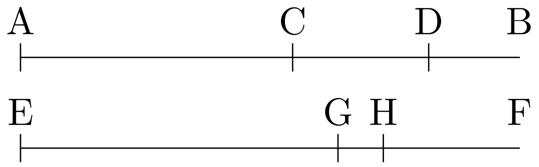
[66.] THE stone in this last instance then not only moves with a velocity perpetually varying, as in the preceeding example, but this variation continually changes. In the true theory of falling bodies, neither this last variation nor any subsequent one can ever be uniform, so that fluxions of every order do here actually exist.

[67.] THE same is true of the motion of the planets in their elliptic orbs; of the motion of light at the confines of different mediums, and of the motion of all pendulous bodies.

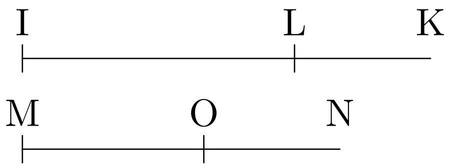
[68.] IN short, an uniform unchangeable velocity is not to be met with in any of those bodies, that fall under our cognisance; for in order to continue such a motion as this, it is necessary, that they should not be disturbed by any force whatever, either of impulse or resistance; but we know of no spaces, in which at least one of these causes of variation does not operate.

[69.] HAVING thus explained the general conception of second, third, and following fluxions; and having shewn, that they are applicable to the circumstances, which do really occur in all motion we are acquainted with; we will now endeavour to declare the manner of assigning them.

[70.] And in the first place second fluxions may be compared together, as follows. Suppose any line to be so described by motion, that it always preserve the same analogy to the first fluxion of any magnitude; then the velocity, wherewith this line is described, that is, the fluxion of this line, will be analogous to the second fluxion of the aforesaid magnitude. For it is evident, that this line will perpetually alter in magnitude in the same proportion, as the fluxion, to which it is analogous, varies.



[71.] Suppose A B to be a fluent described with a varying motion; the second fluxion at any one point C may be compared with the second fluxion at any other point D, by causing the line E F to be described by the motion of a point, so as to keep always the same analogy to the first fluxion of the fluent A B. Suppose E G be to E H, as the first fluxion at C to the first fluxion at D; then the second fluxion at C will be to the second fluxion at D, as the first fluxion of the line E F at G, to the first fluxion of the same at H.

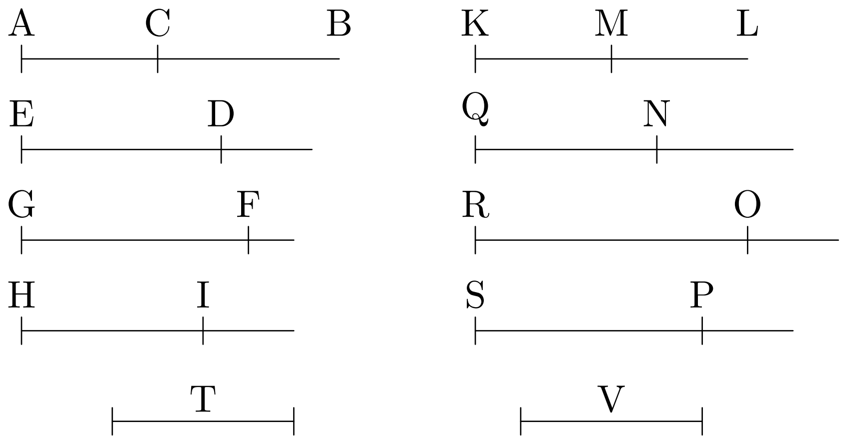


[72.] In like manner, if another fluent I K be generated along with the former fluent A B, and also described with a variable motion; the second fluxion of this latter fluent I K at any place L may be compared with the second fluxion at any part of the former fluent A B, by describing the line M N with such a motion, as always to preserve the same analogy to the first fluxion of the fluent I K, as the line E F bore to the first fluxion of A B. Suppose M O to be to E G, as the first fluxion of I K at L to the first fluxion of A B at C; then the second fluxion at L will be to the second fluxion at C, as the velocity, wherewith the line M N is described at O, to the velocity, wherewith the line E F is described at G.

[73.] In the same manner if a line be described analogous to the second fluxion of any magnitude, the fluxion of this line will express the third fluxion of that magnitude, and so of all the other orders of fluxions.

[74.] In the next place the relation, in which the several orders of fluxions stand with regard to each other, will appear by the following proposition.

[75.] Let the line A B be described by the motion of the point C moving with a varying velocity, and let a series of lines be adapted to this line A B in such manner, that the point D, moving upon the first line of the series at the same time with the point C, may ever terminate a line E D analogous to the velocity of the point C; the point F at the same time terminating upon the second line of this series a line G F analogous to the velocity of the point D; and H I upon the third line being by the motion of the point I made ever analogous to the velocity of the point F; &c.



[76.] If now another line K L be described by the motion of the point M, and if a series of lines be adapted to this line K L in the like analogy by the motion of the points N, O, P, so that Q N be to E D as the velocity of the point M to the velocity of the point C, R O to G F as the velocity of the point N to that of the point D, and S P to H I as the velocity of the point O to that of F; I say, that if the velocity of the point C has to the velocity of the point M always the same proportion at equal distances from A and K, that then the velocity of D to that of N will be in the duplicate of that proportion; the velocity of F to that of O in the triplicate of that proportion; the velocity of I to that of P in the quadruplicate of that proportion, and so on in the same order, as far as these series of lines are extended.

[77.] Suppose the velocity of the point C be always to the velocity of the point M, as the line T to the line V, when these points are at equal distances from A and K. Then, since the times, in which equal lines are described, are reciprocally as the velocities of the describing points; the time, in which A C receives any additional increment, will be to the time, in which K M shall have received an equal increment, as V to T.

[78.] Now E D is always to Q N in the proportion of T to V. Therefore the variation, by increase or diminution, that E D shall receive to the like variation, which Q N shall receive, while the lines A C, K M are augmented by equal increments, will be also as T to V. But the time, wherein E D will receive that variation, to the time, wherein Q N will receive its variation, will be as V to T. Consequently, since the velocities, wherewith different lines are described, are as the lines themselves directly, and as the times of description, reciprocally, the velocity of the point D to that of the point N will be in the duplicate ratio of T to V.

[79.] Again, the velocity of D being to the velocity of N, when A C and K M are equal, always in the same duplicate ratio of T to V, and G F being always to R O as the velocity of the point D to the velocity of the point N, the variation, by increase or diminution, of the line G F to the like variation of R O, while A C and K M receive equal augmentation, will also be as the velocity of D to the velocity of N, that is in the duplicate ratio of T to V. But the time, in which the line G F receives its variation, will be to the time, in which R O receives its variation, as V to T. Hence the velocity of the point F will be to the velocity of the point O in the triplicate ratio of T to V.

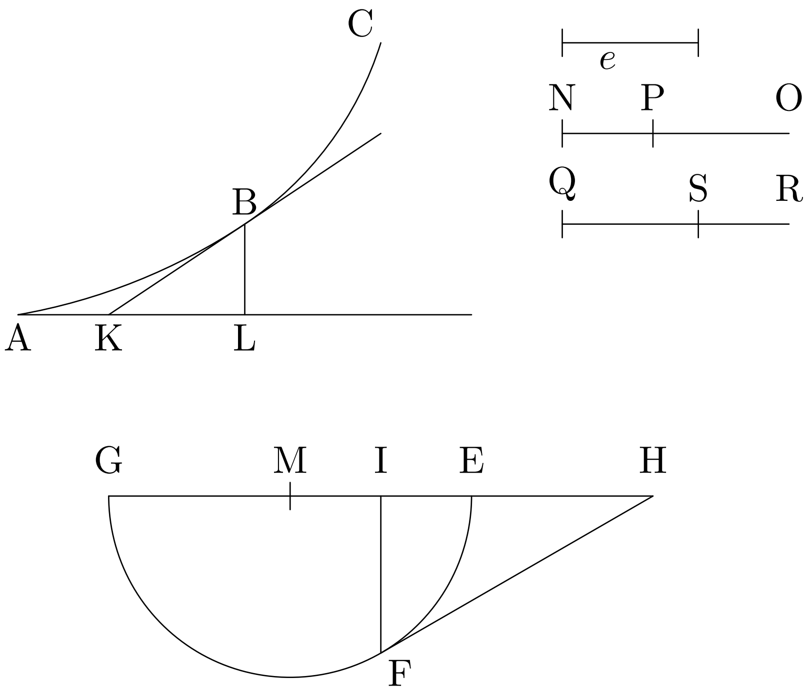
[80.] After the same manner, the velocity of the point I will appear to have to the velocity of the point P the quadruplicate of the ratio of T to V.

[81.] But from what we have said above, it is evident, that the velocity of the point D is to the velocity of the point N, as the second fluxion of A C to the second fluxion of K M; the velocity of the point F to the velocity of the point O, as the third fluxion of A C to the third fluxion of K M; and the velocity of the point I to the velocity of the point P, as the fourth fluxion of A C to the fourth fluxion of K M. And hence appears the truth of Sir Isaac Newton's observation at the end of the first proposition of his book of Quadratures, that a second fluxion, and the second power of a first fluxion, or the product under two first fluxions; a third fluxion, and the third power of a first, or the product under a first and second, and so on; are homologous terms in any equation. For, as it appears by this proposition, that if the velocity, wherewith any fluent is augmented, be in any proportion increased; its second fluxion will increase in the duplicate of that proportion, the third fluxion in the triplicate, and the fourth fluxion in the quadruplicate of that same proportion; it is manifest, that the terms in any equation, that shall involve a second fluxion, will preserve always the same proportion to the terms involving the second power of a first fluxion, or the product of two first fluxions; the terms involving a third fluxion will preserve the same proportion to the terms involving the third power of a first, or the product of a first and second, or the product of three first fluxions; and the terms containing a fourth fluxion will keep the same proportion to the terms containing the fourth power of a first, the product of a second and the second power of a first, the second power of a second, or the product of a first and third; &c. however be increased or diminished the first fluxion, or the velocity, wherewith the fluents augment.

[82.] In the problems concerning curve lines, which relate to the degree of curvature in any point of those curves, or to the variation of their curvature in different parts, these superior orders of fluxions are useful; for by the inflexion of the curve, whilst its abscisse flows uniformly, the fluxion of the ordinate must continually vary, and thereby will be attended with these superior orders of fluxions.

[83.] For example, were it required to compare the different degrees of curvature either of different curves, or of the same curve in different parts, and in order thereto a circle should be sought, whose degree of curvature might be the same with that of any curve proposed, in any point, that should be assigned; such a circle may be found by the help of second fluxions. When the abscisses of two curves flow with equal velocity; where the ordinates have equal first fluxions, the tangents make equal angles with their respective ordinates. If now the second fluxions of these ordinates are also equal, the curves in those points must be equally deflected from their tangents, that is, have equal degrees of curvature. Upon this principle such circles, as have here been mentioned, may be found by the following method.

[84.] The curve A B C being given, let it be required to find a circle equally incurvated with this curve at the point B. Suppose E F G to be this circle, in which the tangent F H at the point F makes with the ordinate F I the same angle, as the tangent B K, drawn to the other curve A B C at the point B, makes with the ordinate B L of that curve. Now if the two abscisses A L and E I are described with equal velocities, the first fluxion of the ordinates L B and I F will be equal; and therefore, if the two curves are equally incurvated at the points B and F, the second fluxions of these ordinates will be also equal. If M be the center of the circle E F G, and M E be denoted by a and M I by x, I F will be $= \sqrt{aa - xx}$; and, by the rules for finding fluxions, the first fluxion of I F will be to the fluxion of M I, or of x, as x to $\sqrt{aa - xx}$.



[85.] Now suppose the line N O to be so described, that the fluxion of M I, or of x, shall be to the first fluxion of I F, as some given line e to N P in the line N O, then will N P be $= \dfrac{ex}{\sqrt{aa - xx}}$. Suppose likewise the line Q R to be so described, that the fluxion of A L in the curve A B C shall be to the first fluxion of L B, as the same given line e to Q S in the line Q R. Here the first fluxions of I F and L B being equal, N P and Q S are equal. And since the second fluxions of I F and L B are equal, the fluxions of N P and Q S are also equal. But N P was $= \dfrac{ex}{\sqrt{aa - xx}}$, and by the rules for finding fluxions, the fluxion of N P will be to the fluxion of M I as eaa to $\overline{aa - xx}\rceil^{\frac{3}{2}}$, that is, as $e \times$ E M q to I F c. Therefore in the curve A B C the fluxion of Q S to the fluxion of A L will be in the same proportion of $e \times$ E M q to I F c. Hence by finding first Q S, then its fluxion, from the equation expressing the nature of the curve A B C, the proportion of $e \times$ E M q to I F c will be given. Consequently the proportion of e to I F will be also given, because the ratio of E M q to I F q is the same with the given ratio of H F q to H I q, or of K B q to K L q. And hereby the circle E F G will be given, whose curvature is equal to the curvature of the curve A B C at the point B.

[86.] Suppose the curve A B C to be the conical parabola, where A L q shall be equal to $\gamma \times$ L B, γ being the latus rectum of the axis. Here e will be to Q S as γ to 2 A L; for that is the ratio of the fluxion of A L to the fluxion of B L: therefore Q S is $= \dfrac{2e}{\gamma}$ A L; and consequently the fluxion of Q S to the fluxion of A L (that is $e \times$ E M q to I F c) as $2e$ to γ, or as $2e \times$ E M q to $\gamma \times$ E M q; in so much that I F c is $= \frac{1}{2}\gamma \times$ E M q, and the given ratio of I F q to E M q (namely the ratio of K L q to K B q) is the same with the ratio of $\frac{1}{2}\gamma$ to I F: that is, I F is equal to half the latus rectum appertaining to the diameter of the parabola, whose vertex is the point B.

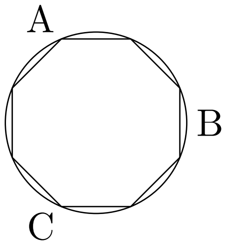
[87.] This is all we think necessary towards giving a just and clear idea of the nature of fluxions, and for proving the certainty of the deductions made from them. For it must now be manifest to every reader, that mathematical quantities become the proper object of this doctrine of fluxions, whenever they are supposed to increase by any continued motion of prolongation, dilation, expansion or other kinds of augmentation, provided such augmentation be directed by some general rule, whence the measure of the increase of these quantities may from time to time be estimated. And when different homogeneous magnitudes increase after this manner together, one may vary faster than another. Now the velocity of increase in each quantity, is the fluxion of that quantity. This is the true interpretation of Sir Isaac Newton's appellation of fluxions, Incrementorum velocitates. For this doctrine does not suppose the fluents themselves to have any motion. Fluxions are not the velocities, with which the fluents, or even the increments, which those fluents receive, are themselves moved; but the degrees of velocity, wherewith those increments are generated. Subjects incapable of local motion, such as fluxions themselves, may also have their fluxions. In this we do not ascribe to these fluxions any actual motion; for to ascribe motion, or velocity to what is itself only a velocity, would be wholly unintelligible. The fluxion of another fluxion is only a variation in the velocity, which is that fluxion. In short, light, heat, sound, the motion of bodies, the power of gravity, and whatever else is capable of variation, and of having that variation assigned, for this reason comes under the present doctrine; nothing more being understood by the fluxion of any subject, than the degree of such its variation.

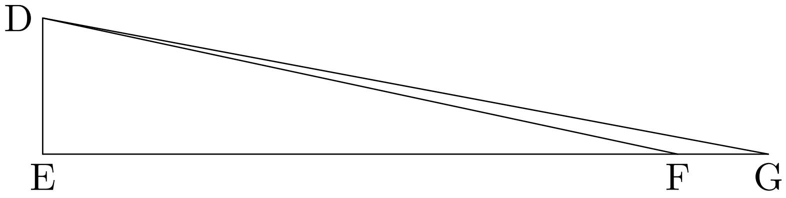
[88.] To assign the velocities of variation or increase in different homogeneous quantities, it is necessary to compare the degrees of augmentation, which those quantities receive in equal portions of time; and in this doctrine of fluxions no further use is made of such increments: for the application of this doctrine to geometrical problems depends upon the knowledge of these velocities only. But the consideration of the increments themselves may be made subservient to the like uses upon other principles; the explanation of which leads us to the second part of our design.

OF

PRIME and ULTIMATE RATIOS.

[89.] The primary method of comparing together the magnitudes of rectilinear spaces is by laying them one upon another: thus all the right lined spaces, which in the first book of Euclide are proved to be equal, are the sum or difference of such spaces, as would cover one another. This method cannot be applied in comparing curvilinear spaces with rectilinear ones; because no part whatever of a curve line can be laid upon a streight line, so as wholly to coincide with it. For this purpose therefore the ancient geometers made use of a method of reasoning, since commonly called the method of exhaustions; which consists in describing upon the curvilinear space a rectilinear one, which though not equal to the other, yet might differ less from it than by any the most minute difference whatever, that should be proposed; and thereby proving, the two spaces, they would compare, could be neither greater nor less than each other.





[90.] For example, in order to prove the equality between the space, comprehended within the circumference of a circle, and a triangle, whose base should be equal to the circumference of that circle, and its altitude to the semidiameter, Archimedes takes this method. About the circle he describes a polygon as A B C, and makes it appear, that by multiplying the sides of this polygon, there may at length be described such a one, as shall exceed the circle less than by any difference, that shall be proposed, how minute soever that difference be. By this means it was easy to prove, that the triangle D E F, whose base E F should be equal to the circumference of the circle, and altitude E D equal to the semidiameter, is not greater than the circle. For were it greater, how small soever be the excess, it were possible to describe about the circle a polygon less than the triangle; but the circumference of the polygon is greater than the circumference of the circle, therefore the polygon can never be less, but must be always greater than the triangle; for the polygon is equal to a triangle, whose altitude is the semidiameter of the circle, and base equal to the circumference of the polygon. It appears therefore impossible for the triangle D E F to be greater than the circle.

[91.] THUS far Archimedes makes use of the polygon circumscribing the circle and no farther: but inscribing another within the circle he proves, by a similar process of reasoning, that it is impossible for the triangle to be less than the circle; whereby at length it becomes certain, that the triangle D E F is neither greater nor less than the circle, but equal to it.

[92.] HOWEVER, the triangle may be proved not to be less than the circle by the circumscribed polygon also. For were it less, another triangle D E G, whose base E G is greater than E F, might be taken, which should not be greater than the circle. But a polygon can be circumscribed about the circle, the circumference of which shall exceed the circumference of the circle by less than any line, that can be named, consequently by less than F G, that is, the circumference of the polygon shall be less than E G, and the polygon less than the triangle D E G; therefore it is impossible, that this triangle should not exceed the circle, since it is greater than the polygon: consequently the triangle D E F cannot be less than the circle.

[93.] THUS the circle and triangle may be proved to be equal by comparing them with one polygon only, and Sir Isaac Newton has instituted upon this principle a briefer method of conception and expression for demonstrating this sort of propositions, than what was used by the ancients; and his ideas are equally distinct, and adequate to the subject, with theirs, though more complex. It became the first writers to choose the most simple form of expression, and the least compounded ideas possible. But Sir Isaac Newton thought, he should oblige the mathematicians by using brevity, provided he introduced no modes of conception difficult to be comprehended by those, who are not unskilled in the ancient methods of writing.

[94.] THE concise form, into which Sir Isaac Newton has cast his demonstrations, may very possibly create a difficulty of apprehension in the minds of some unexercised in these subjects. But otherwise his method of demonstrating by the prime and ultimate ratios of varying magnitudes is not only just, and free from any defect in itself; but easily to be comprehended, at least by those who have made these subjects familiar to them by reading the ancients.

[95.] IN this method any fix'd quantity, which some varying quantity, by a continual augmentation or diminution, shall perpetually approach, but never pass, is considered as the quantity, to which the varying quantity will at last or ultimately become equal; provided the varying quantity can be made in its approach to the other to differ from it by less than by any quantity how minute soever, that can be assigned*.

[96.] UPON this principle the equality between the fore-mentioned circle and triangle D E F is at once deducible. For since the polygon circumscribing the circle approaches to

* Princ. Philos. Lib. I. Lem. I.

each according to all the conditions above set down, this polygon is to be considered as ultimately becoming equal to both, and consequently they must be esteemed equal to each other.

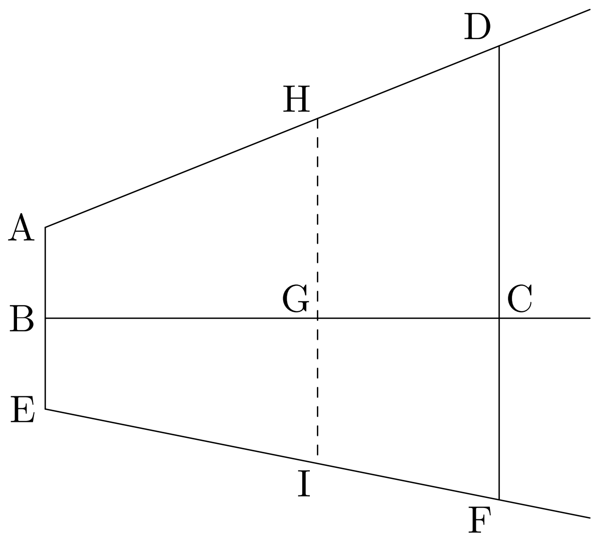
[97.] THAT this is a just conclusion, is most evident. For if either of these magnitudes be supposed less than the other, this polygon could not approach to the least within some assignable distance.

[98.] RATIOS also may vary, as to be confined after the same manner to some determined limit, and such limit of any ratio is here considered as that, with which the varying ratio will ultimately coincide*.

[99.] FROM any ratio's having such a limit, it does not follow, that the variable quantities exhibiting that ratio have any final magnitude, or even limit, which they cannot pass.

[100.] FOR suppose two magnitudes, B and B + A, whose difference shall be A, are each of them perpetually increasing by equal degrees. It is evident, that if A remains unchanged, the proportion of B + A to B is a proportion, that tends nearer and nearer to the proportion of equality, as B becomes larger; it is also evident, that the proportion of B + A to B may, by taking B of a sufficient magnitude, be brought at least nearer to the proportion of equality, than to any other assignable proportion; and consequently the ratio of equality is to be considered as the ultimate ratio of B + A to B. The ultimate proportion then of these quantities is here assigned, though the quantities themselves have no final magnitude.

[101.] THE same holds true in decreasing quantities.



[102.] THE quadrilateral A B C D bears to the quadrilateral E B C F the proportion of A B + D C to B E + C F, provided the two lines A E and D F are parallel. Now if the line D F be drawn nearer to A E, this proportion of A B + D C to B E + C F will not remain the same, unless the lines D A, C B, F E produced will meet in the same point; and this

* Ibid.

proportion, by diminishing the distance between D F and A E may at last be brought nearer to the proportion of A B to B E, than to any other whatever. Therefore the proportion of A B to B E is to be considered as the ultimate proportion of A B + D C to B E + C F, or as the ultimate proportion of the quadrilateral A B C D to the quadrilateral E B C F.

[103.] HERE these quadrilaterals can never bear one to the other the proportion between A B and B E, nor have either of them any final magnitude, or even so much as a limit, but by the diminution of the distance between D F and A E they diminish continually without end: and the proportion between A B and B E is for this reason called the ultimate proportion of the two quadrilaterals, because it is the proportion, which those quadrilaterals can never actually have to each other, but the limit of that proportion.

[104.] THE quadrilaterals may be continually diminished, either by dividing B C in any known proportion in G drawing H G I parallel to A E, by dividing again B G in the like manner, and by continuing this division without end; or else the line D F may be supposed to advance towards A E with an uninterrupted motion, 'till the quadrilaterals quite disappear, or vanish. And under this latter notion these quadrilaterals may very properly be called vanishing quantities, since they are now considered, as never having any stable magnitude, but decreasing by a continued motion, 'till they come to nothing. And since the ratio of the quadrilateral A B C D to the quadrilateral B E F C, while the quadrilaterals diminish, approaches to that of A B to B E in such manner, that this ratio of A B to B E is the nearest limit, that can be assigned to the other; it is by no means a forced conception to consider the ratio of A B to B E under the notion of the ratio, wherewith the quadrilaterals vanish; and this ratio may properly be called the ultimate ratio of two vanishing quantities.

[105.] THE reader will perceive, I am endeavouring to explain Sir Isaac Newton's expression Ratio ultima quantitatum evanescentium; and I have rendered the Latin participle evanescens, by the English one vanishing, and not by the word evanescent; which having the form of a noun adjective, does not so certainly imply that motion, which ought here to be kept carefully in mind. The quadrilaterals A B C D, B E F C become vanishing quantities from the time, we first ascribe to them this perpetual diminution; that is, from that time they are quantities going to vanish. And as during their diminution they have continually different proportions to each other; so the ratio betweeen A B and B E is not to be called merely Ratio harum quantitatum evanescentium, but Ultima ratio*.

[106.] SHOULD we suppose the line D F first to coincide with the line A E, and then recede from it, by that means giving birth to the quadrilaterals; under this conception the ratio of A B to B E may very justly be considered as the ratio, wherewith the quadrilaterals by this motion commence; and this ratio may also properly be called the first or prime ratio of these quadrilaterals at their origine.

[107.] HERE I have attempted to explain in like manner the phrase Ratio prima quantitatum nascentium; but no English participle occuring to me, whereby to render the word nascens, I have been obliged to use circumlocution. Under the present conception of the quadrilaterals they are to be called nascentes, not only at the very instant of their first production, but according to the sense, in which such participles are used in common speech,

* Vid. Princ. Philos. pag. 37, 38.

after the same manner, as when we say of a body, which has lain at rest, that it is beginning to move, though it may have been some little time in motion: on this account we must not use the simple expression Ratio quantiatum nascentium; for by this we shall not specify any particular ratio; but to denote the ratio between A B and B E we must call it Ratio prima quadrilaterum nascentium*.

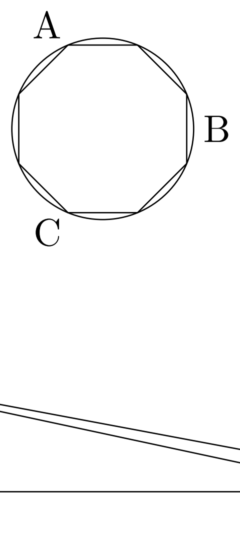
[108.] We see here the same ratio may be called sometimes the prime, at other times the ultimate ratio of the same varying quantities, as these quantities are considered either under the notion of vanishing, or of being produced before the imagination by an uninterrupted motion. The doctrine under examination receives its name from both these ways of expression.

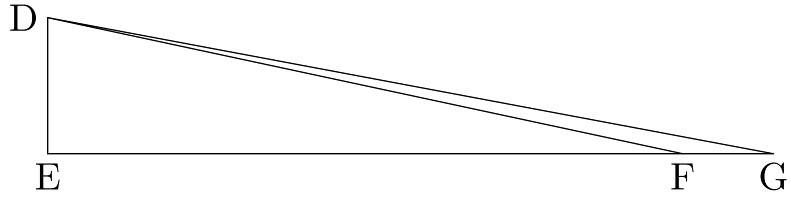
[109.] Thus we have given a general idea of the manner of conception, upon which this doctrine is built. But as in the former part of this discourse we confirmed the doctrine of fluxions by demonstrations of the most circumstantial kind; so here, to remove all distrust concerning the conclusiveness of this method of reasoning, we shall draw out its first principles into a more diffusive form.

[110.] For this purpose, we shall in the first place define an ultimate magnitude to be the limit, to which a varying magnitude can approach within any degree of nearness whatever, though it can never be made absolutely equal to it.

[111.] Thus the circle discoursed of above, according to this definition, is to be called the ultimate magnitude of the polygon circumscribing it; because this polygon, by increasing the number of its sides, can be made to differ from the circle, less than by any space, that can be proposed how small soever; and yet the polygon can never become either equal to the circle or less.

[112.] In like manner the circle will be the ultimate magnitude of the polygon inscribed, with this difference only, that as in the first case the varying magnitude is always greater, here it will be less than the ultimate magnitude, which is its limit.





* Vid. Ibid.

[113.] Again the triangle D E F is the ultimate magnitude of the triangle D E G; because the base E G, being always equal to the circumference of the polygon, will constantly be greater than the base E F, equal to the circumference of the circle only, and yet E G may be made to approach E F nearer than by any difference, that can be named.

[114.] Upon this definition we may ground the following proposition; That, when varying magnitudes keep constantly the same proportion to each other, their ultimate magnitudes are in the same proportion.

[115.] Let A and B be two varying magnitudes, which keep constantly in the same proportion to each other; and let C be the ultimate magnitude of A, and D the ultimate magnitude of B. I say that C is to D in the same proportion.

A. B.

C. D.

E.

[116.] As A is a varying magnitude continually approaching to C, but can never become equal to it, A may be either always greater than or always less than C. In the first place suppose it greater. When A is greater than C, in approaching to C it is constantly diminished; therefore B keeping always in the same proportion to A, B in approaching to its limit D is also continually diminished.

[117.] Now, if possible, let the ratio of C to D be greater than that of A to B, that is, suppose C to have to some magnitude E, greater than D, the same proportion as A has to B. Since C is the ultimate magnitude of A in the sense of the preceeding definition, A can be made to approach nearer to C than by any difference, that can be proposed, but can never become equal to it, or less. Therefore, since C is to E as A to B, B will always exceed E; consequently can never approach to D so near as the excess of E above D: which is absurd. For since D is supposed the ultimate magnitude of B, it can be approached by B nearer than by any assignable difference.

[118.] After the same manner, neither can the ratio of D to C be greater than that of B to A.

[119.] If the varying magnitude A be less than C, it follows, in like manner, that neither the ratio of C to D can be less than that of A to B, nor the ratio of D to C less than that of B to A.

[120.] It is an evident corollary from this proposition, that the ultimate magnitudes of the same or equal varying magnitudes are equal.

[121.] Now from this proposition the fore-mentioned equality between the circle and triangle D E F will again readily appear. For the circle being the ultimate magnitude of the polygon, and the triangle D E F the ultimate magnitude of the triangle D E G; since the polygon and the triangle D E G are equal, by this proposition the circle and triangle D E F will be also equal.

[122.] IF the preceeding proposition be admitted, as a genuine deduction from the definition, upon which it is grounded; this demonstration of the equality of the circle and triangle cannot be excepted to. For as no objection can lie against the definition itself, as no inference is there deduced, but only the sense explained of the term defined in it.

[123.] THE other part of this method, which concerns varying ratios, may be put into the same form by defining ultimate ratios as follows.

[124.] IF there be two quantities, that are (one or both) continually varying, either by being continually augmented, or continually diminished; though the proportion, they bear to each other, should by this means perpetually vary, but in such a manner, that it constantly approaches nearer and nearer to some determined proportion, and can also be brought at last in its approach nearer to this determined proportion than to any other, that can be assigned, but can never pass it: this determined proportion is then called the ultimate proportion, or the ultimate ratio of those varying quantities.

[125.] TO this definition of the sense, in which the term ultimate ratio, or ultimate proportion is to be understood, we must subjoin the following proposition: That all the ultimate ratios of the same varying ratio are the same with each other.

[126.] SUPPOSE the ratio of A to B continually varies by the variation of one or both of the terms A and B. If the ratio of C to D be the ultimate ratio of A to B, and the ratio of E to F be likewise the ultimate ratio of the same; I say, the ratio of C to D is the same with the ratio of E to F.

[127.] IF possible, let the ratio of E to F differ from that of C to D. Since the ratio of C to D is the ultimate ratio of A to B, the ratio of A to B, in its approach to that of C to D, can be brought at last nearer to it, than to any other whatever. Therefore if the ratio of E to F differ from that of C to D, the ratio of A to B will either pass that of E to F, or can never approach so near to it, as to the ratio of C to D: in so much that the ratio of E to F cannot be the ultimate ratio of A to B, in the sense of this definition.

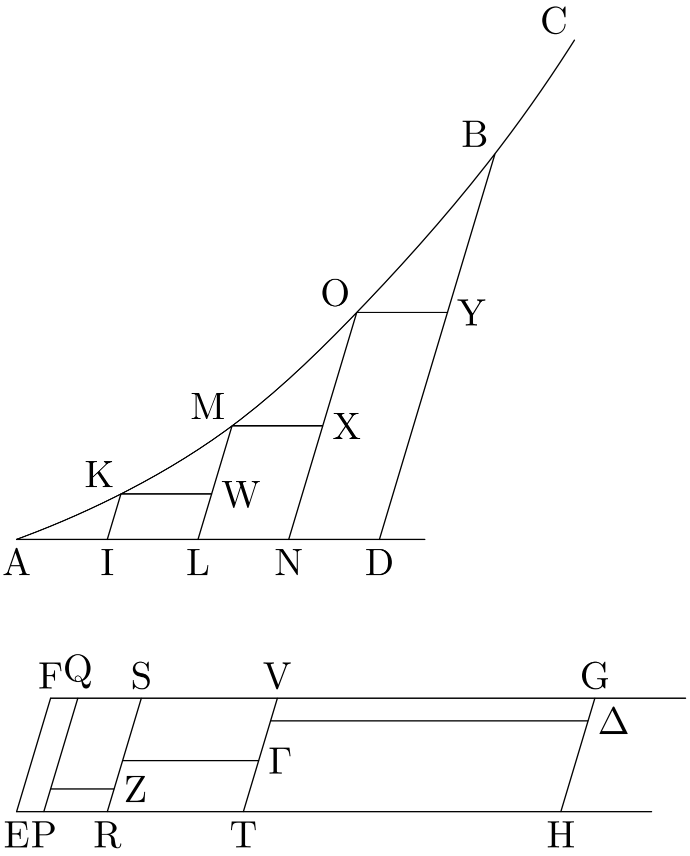
[128.] THE two definitions here set down, together with the general propositions annexed to them, comprehend the whole foundation of this method, we are now explaining.

[129.] WE find in former writers some attempts toward so much of this method, as depends upon the first definition. Lucas Valerius in a most excellent treatise on the Center of gravity of solid bodies, has given a proposition nothing different, but in the form of the expression, from that we have subjoined to our first definition; from which he demonstates with brevity and elegance his propositions concerning the mensuration and center of gravity of the sphere, spheroid, parabolical and hyperbolical conoids. This author writ before the doctrine of indivisibles was proposed to the world. And since, Andrew Tacquet, in his treatise on the Cylindrical and annular solids, has made the same proposition, though something differently expressed, the basis of his demonstrations at the same time very judiciously exposing the inconclusiveness of the reasonings from indivisibles. However, the consideration of the limits of varying proportions, when the quantities expressing those proportions have themselves no limits, which renders this method of prime and ultimate ratios much more extensive, we owe intirely to Sir Isaac Newton. That this method, as thus compleated, is applicable not only to the subjects treated by the ancients in the method of exhaustions, but to many others also of the greatest importance, appears from our author's immortal treatise on the Mathematical principles of natural philosophy.

[130.] HOWEVER, we shall farther illustrate this doctrine in applying it to the same general problems as those, whereby the use of fluxions was above exemplified.

[131.] WE have already given one instance of its use in determining the dimensions of curvilinear spaces; we shall now set forth the same by a more general example.

[132.] LET A B C be a curve line, its abscisse A D, and an ordinate D B. If the parallelogram E F G H, formed upon the given line E F under the same angle, as the ordinates of the curve make with its abscisse, be in all parts so related to the curve, that the ultimate ratio of any portion of the abscisse A D to the correspondent portion of the line E H, shall be that of the given line E F to the ordinate of the curve at the beginning of that portion of the abscisse; then will the curvilinear space A D B be equal to the parallelogram E G.



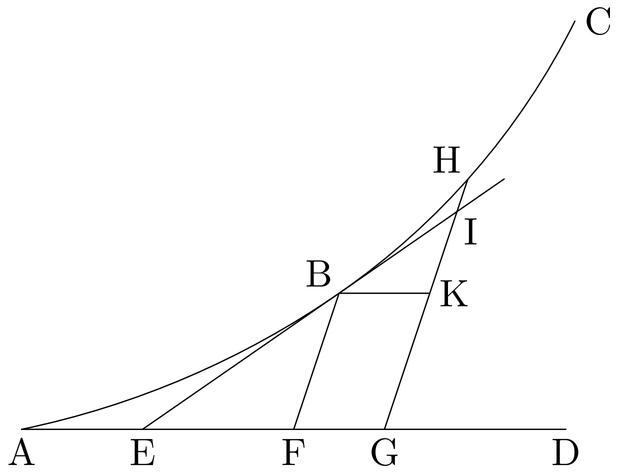
[133.] IN the curve let the abscisse A D be divided into any number of equal parts A I, I L, L N, N D, and let the ordinates I K, L M, N O be drawn, and also in the parallelogram E G the correspondent lines P Q, R S and T V. In the curve compleat the parallelograms I W, L X, N Y, and in the parallelogram E G make the parallelogram P Z equal to the parallelogram I W, the parallelogram R Γ equal to L X, and the parallelogram T Δ equal to N Y: then the whole figure I K W M X O Y D will be equal to the whole Figure P Z Γ Δ H. But in the curve, by increasing the number and diminishing the breadth of these parallelograms, the figure I K W M X O Y D will approach nearer and nearer in magnitude to the curvilinear space A D B; in so much that their difference may be reduced to less than any space, that shall be assigned; therefore the curvilinear space A D B is the ultimate magnitude of the figure I K W M X O Y D. Farther, since the parallelogram E G is in all parts so related to the curve, that the ultimate ratio of every portion, as L N, of the abscisse A D to R T, the correspondent portion of E H, is the same with the ratio of E F or R S, to L M; the ultimate ratio of the parallelogram L X, or its equal R Γ, to the parallelogram R V, is the ratio of equality. This is also true of all the other correspondent parallelograms; therefore, the ultimate ratio of the figure P Z Γ Δ H to the parallelogram P G is the ratio of equality; that is the figure P Z Γ Δ H, by increasing the number of its parallelograms, can be brought nearer to the parallelogram P G than by any difference whatever, that may be proposed. Moreover, by increasing of the number of ordinates in the curve, the residuary portion A I of the abscisse can be reduced to less than any magnitude, that shall be proposed; whereby the parallelogram E Q, corresponding to this portion of the abscisse, may be also reduced to less than any magnitude, that can be proposed; and the parallelogram P G be brought to differ less from E G than by any assignable magnitude. Since therefore the figure P Z Γ Δ H can be brought nearer to the parallelogram P G than by any difference, that can be assigned; the Figure P Z Γ Δ H can be brought also nearer to the parallelogram E G than by any difference, that can be assigned. Consequently the parallelogram E G is the ultimate magnitude of the figure P Z Γ Δ H. Therefore the figures P Z Γ Δ H and I K W M X O Y D being equal varying magnitudes, and the ultimate magnitudes of equal varying magnitudes being equal, the curvilinear space A D B is equal to the parallelogram E G.

[134.] Suppose the curve A B C were a cubical parabola convex to the abscisse, that is, suppose Θ a given line, and $\Theta\,q \times \mathrm{L\,M} = \mathrm{A\,L}\,c$. If E H be $= \dfrac{\mathrm{A\,D}\,qq}{4\Theta\,q \times \mathrm{E\,F}}$, then the parallelogram E G will be equal to the space A D B.

[135.] As E H is $= \dfrac{\mathrm{A\,D}\,qq}{4\Theta\,q \times \mathrm{E\,F}}$, E R will be $= \dfrac{\mathrm{A\,L}\,qq}{4\Theta\,q \times \mathrm{E\,F}}$ and E T $= \dfrac{\mathrm{A\,N}\,qq}{4\Theta\,q \times \mathrm{E\,F}}$, consequently

$$\mathrm{R\,T} = \frac{\mathrm{A\,L}\,c \times \mathrm{L\,N} + \frac{3}{2}\,\mathrm{A\,L}\,q \times \mathrm{L\,N}\,q + \mathrm{A\,L} \times \mathrm{L\,N}\,c + \frac{1}{4}\,\mathrm{L\,N}\,qq}{\Theta\,q \times \mathrm{E\,F}}.$$

Therefore the parallelogram E G is here so related in all parts to the curve, that L N is to R T as $\Theta\,q \times \mathrm{E\,F}$ to $\mathrm{A\,L}\,c + \frac{3}{2}\,\mathrm{A\,L}\,q \times \mathrm{L\,N} + \mathrm{A\,L} \times \mathrm{L\,N}\,q + \frac{1}{4}\,\mathrm{L\,N}\,c$. Now it is evident, that the ratio of L N to R T can never be so great as the ratio of $\Theta\,q \times \mathrm{E\,F}$ to $\mathrm{A\,L}\,c$; but yet, by diminishing L N, the ratio of L N to R T may at last be brought nearer to this ratio than to any other whatever, than should be proposed. Consequently by the preceeding definition of what is to be understood by an ultimate ratio, the ratio of $\Theta\,q \times \mathrm{E\,F}$ to $\mathrm{A\,L}\,c$ is the ultimate ratio of L N to R T. But $\mathrm{A\,L}\,c$ being $= \Theta\,q \times \mathrm{L\,M}$, $\Theta\,q \times \mathrm{E\,F}$ is to $\mathrm{A\,L}\,c$ as E F to L M. Therefore the ratio of E F to L M is the ultimate ratio of L N to R T. Consequently, by the preceeding general proposition, the parallelogram E G is equal to the curvilinear space A D B. And this parallelogram is equal to $\frac{1}{4}\,\mathrm{A\,D} \times \mathrm{D\,B}$.



[136.] Again this method is equally useful in determining the situation of the tangents to curve lines.

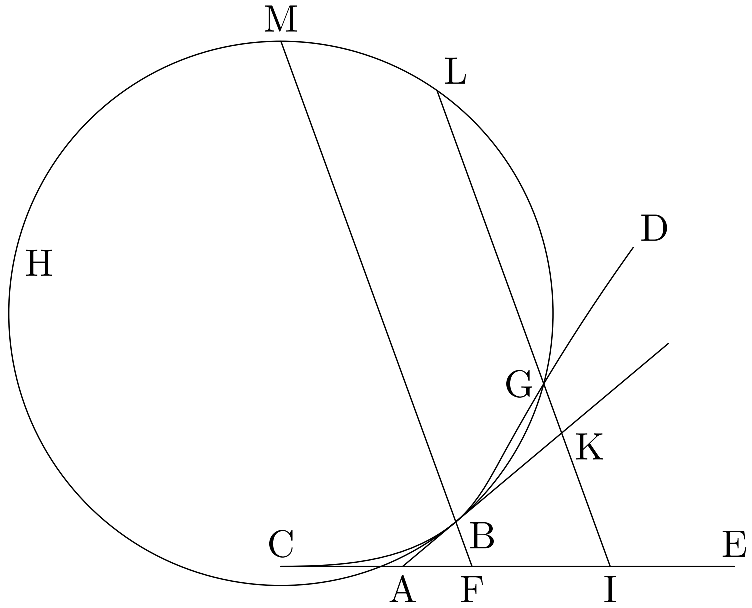
[137.] In the curve A B C, whose abscisse is A D, let E B be a tangent at the point B. Let B F be the ordinate at the same point B, and G H another ordinate parallel to it, which shall meet the tangent in I, and the line B K, parallel to the abscisse A D, in K. Here the ratio of H K, the difference of the ordinates, to B K can never be the same with the ratio of B F to F E, unless by the figure of the curve the tangent chance to cut it in some point remote from B; this ratio of B F to F E being the same with that of I K to K B. But it is farther evident, that the nearer G H is to F B, the ratio of K H to K B will approach so much the nearer to the ratio of I K to K B; and the angle, which the curve B C makes with the tangent B I being less than any right-lined angle, it is manifest, that G H may be made to approach towards F B, 'till the ratio of H K to K B, shall at last approach nearer to the ratio of I K to K B, or of B F to F E, than to any other ratio whatever, that shall be proposed; that is, the ratio of B F to F E is the ultimate ratio of H K to K B. Therefore, if from the properties of the curve A B C the ratio of H K to K B be determined, and from thence their ultimate ratio assigned; this ratio thus assigned will be the ratio of B F to F E; because all the ultimate ratios of the same variable ratio are the same with each other.

[138.] Suppose the curve A B C again to be a cubical parabola, where B F is $= \dfrac{\mathrm{A\,F}\,c}{\mathrm{Z}\,q}$, and G H $= \dfrac{\mathrm{A\,G}\,c}{\mathrm{Z}\,q}$. Here H K will be $= \dfrac{3\,\mathrm{A\,F} \times \mathrm{A\,G} \times \mathrm{F\,G} + \mathrm{F\,G}\,c}{\mathrm{Z}\,q}$; therefore H K is to F G, or B K, as $3\,\mathrm{A\,F} \times \mathrm{A\,G} + \mathrm{F\,G}\,q$ to $\mathrm{Z}\,q$. Consequently the ratio of H K to B K can never be so small as the ratio of $3\,\mathrm{A\,F}\,q$ to $\mathrm{Z}\,q$; but by diminishing B K it may be brought nearer to that ratio, than to any other whatever; that is, the ratio of $3\,\mathrm{A\,F}\,q$ to $\mathrm{Z}\,q$ is the ultimate ratio of H K to K B. Therefore, if B F bear to F E the ratio of $3\,\mathrm{A\,F}\,q$ to $\mathrm{Z}\,q$, the line B E will touch the curve in B: and E F will be equal to $\frac{1}{3}$ A F.

[139.] After the situation of the tangent has been thus determined, the magnitude of H I, the part of the ordinate intercepted between the tangent and the curve, will be known. For example, in this instance since B F is to F E, that is I K to F G, as $3\,\mathrm{A\,F}\,q$ to $\mathrm{Z}\,q$, I K will be $= \dfrac{3\,\mathrm{AF}q \times \mathrm{FG}}{\mathrm{Z}q}$, and HK being $= \dfrac{3\,\mathrm{AF} \times \mathrm{AG} \times \mathrm{FG} + \mathrm{FG}c}{\mathrm{Z}q}$, HI will be

$$= \frac{3\,\mathrm{AF} \times \mathrm{FG}q + \mathrm{FG}c}{\mathrm{Z}q} = \frac{\mathrm{FG}q}{\mathrm{Z}q} \times \overline{3\,\mathrm{AF} + \mathrm{FG}}.$$

Now by this line HI may the curvature of curve lines be compared.



[140.] Let the streight line AB touch the curve CBD in the point B; CE being the abscisse of the curve, and BF the ordinate at B. Take any other point G in the curve, and through the points G, B describe the circle BGH, that shall touch the line AB in B; lastly, draw IKGL parallel to FB. Here are two angles formed at the point B with the circle, one by the line BK, the other by the curve; and the proportion of the first of these angles to the second will be different in different distances of the point G from the point B. And by the approach of G to B the angle between the circle and curve will be diminished, even so much as at length to bear a less proportion to the angle between the circle and tangent, than any, that can be proposed. That is, by the approach of the point G to B the angle between the tangent and circle may be brought nearer to the angle between the tangent and the curve, than by any difference how minute soever homogeneous to those angles; therefore the magnitude of the circle being continually varied by the gradual approach of G to B, and the angle between the tangent and circle thereby also varied; the angle between the tangent and curve is the ultimate magnitude of these angles. That is, the ultimate of these circles determines the degree of curvature of the curve CBD at the point B. But in the circle the rectangle under LKG is equal to the square of BK. And whereas the magnitude of KL will perpetually vary by the approach of the point G towards B; if BM taken in FB produced be the ultimate magnitude of KL, the circle described through M and B to touch the tangent AK in B will be the circle, by which the curvature of the curve CBD in B is to be estimated.

[141.] Suppose the curve CBD to be the cubical parabola as before, where $\mathrm{Z}q \times \mathrm{FB}$ is $= \mathrm{CF}c$, then KG will be $= \dfrac{\mathrm{FI}q}{\mathrm{Z}q} \times \overline{3\,\mathrm{CF} + \mathrm{FI}}$. Hence LK $(= \dfrac{\mathrm{BK}q}{\mathrm{KG}})$ is $= \dfrac{\mathrm{BK}q}{\mathrm{FI}q} \times \dfrac{\mathrm{Z}q}{3\,\mathrm{CF} + \mathrm{FI}}$.

But it is evident, that in a given situation of the tangent AB the ratio of BKq to FIq is given; therefore LK will be reciprocally as $3CF + FI$, and will continually increase, as the point G approaches to the point B, but can never be so great, as to equal $= \frac{BKq}{FIq} \times \frac{Zq}{3CF}$; yet by the near approach of G to B, LK may be brought nearer to this quantity than by any difference, that can be proposed. Therefore, by our former definition of ultimate magnitudes, $\frac{BKq}{FIq} \times \frac{Zq}{3CF}$ is the ultimate magnitude of LK. Consequently, if BM be taken equal to this $\frac{BKq}{FIq} \times \frac{Zq}{3CF}$, the circle described through M is that required.

[142.] We have now gone through all, we think needful for illustrating the doctrine of prime and ultimate ratios; and by the definitions, we have given of ultimate magnitudes and proportions, compared with the instances, we have subjoined, of the application of this doctrine to geometrical problems, we hope our readers cannot fail of forming so distinct a conception of this method of reasoning, that it shall appear to them equally geometrical and scientific with the most unexceptionable demonstration.

[143.] Therefore we shall in the next place proceed to consider the demonstrations, which Sir Isaac Newton has himself given, upon the principles of this method, for his precepts for assigning the fluxions of flowing quantities.

OF

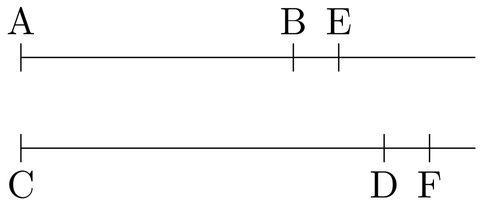
Sir Isaac Newton's

METHOD

Of demonstrating his Rules for finding

FLUXIONS.

[144.] Sir Isaac Newton has comprehended his directions for computing the fluxions of quantities in two propositions; one in his Introduction to his treatise on the Quadrature of curves; the other is the first proposition of the book itself. In the first he assigns the fluxion of a simple power, the latter is universal for all quantities whatever.



[145.] For determining the fluxion of a simple power suppose the line A B to be denoted by x, and another line C D to be denoted by $\frac{x^n}{a^{n-1}}$, or by considering a as unite, C D will be denoted by x^n.

[146.] Suppose the points B and D to move in equal spaces of time into two other positions E and F; then D F will be to B E in the ratio of the velocity, wherewith D F would be described with an uniform motion, to the velocity, wherewith B E will be described in the same time with an uniform motion. But if the point describing the line A B moves uniformly; the velocity, wherewith the line C D is described, will not be uniform. Therefore the space D F is not described with a uniform velocity; in so much that the velocity, wherewith D F would be uniformly described; is never the same with the velocity at the point D. But by diminishing the magnitude of D F, the uniform velocity, wherewith D F would be described, may be made to approach at pleasure to the velocity at the point D. Therefore the velocity at the point D is the ultimate magnitude of the velocity, wherewith D F would be uniformly described. Consequently the ratio of the velocity at D to the velocity at B is the ultimate ratio of the velocity, wherewith D F would be uniformly described, to the velocity, wherewith B E is uniformly described. But D F being to B E as the velocity, wherewith D F would be uniformly described, to that, wherewith B E is uniformly described, the ultimate ratio of D F to B E is also the ultimate ratio of the first of these velocities to the last; because all the ultimate ratios of the same varying ratio are the same with each other. Therefore the ratio of the velocity at D to the velocity at B, that is, of the fluxion of C D to the fluxion of A B, is the same with the ultimate ratio of D F to B E.

[147.] If now the augment B E be denoted by o, the augment D F will be denoted by

$$nx^{n-1}o + \frac{n\times\overline{n-1}}{2}\times x^{n-2}o^2 + \frac{n\times\overline{n-1}\times\overline{n-2}}{6}\times x^{n-3}o^3 + \&c.$$

And here it is obvious, that all the terms after the first taken together may be made less than any assignable part of the first. Consequently the proportion of the first term $nx^{n-1}o$ to the whole augment may be made to approach within any degree whatever of the proportion of equality; and therefore the ultimate proportion of

$$nx^{n-1}o + \frac{n\times\overline{n-1}}{2}\times x^{n-2}o^2 + \frac{n\times\overline{n-1}\times\overline{n-2}}{6}\times x^{n-3}o^3 + \&c.$$

to o, or of D F to B E, is that of $nx^{n-1}o$ only to o, or the proportion of nx^{n-1} to 1.

[148.] And we have already proved, that the proportion of the velocity at D to the velocity at B is the same with the ultimate proportion of D F to B E; therefore the velocity at D is to the velocity at B, or the fluxion of x^n to the fluxion of x as nx^{n-1} to 1.

[149.] In the first proposition of the treatise of Quadratures the author proposes the relation betwixt three varying quantities x, y, and z to be expressed by ths equation $x^3 - xy^2 + a^2z - b^3 = 0$. Suppose these quantities to be augmented by any contemporaneous increments great or small. Let us also suppose some quantity o to be described at the same time by some known velocity, and let that velocity be denoted by m; the velocity, wherewith the augment of x would be uniformly described in that time be denoted by $\dot{x}$; the velocity, wherewith the augment of y would be uniformly described in the same time by $\dot{y}$; and lastly the velocity, wherewith the augment of z would be uniformly described in the same time by $\dot{z}$. Then $\frac{o\dot{x}}{m}$, $\frac{o\dot{y}}{m}$, and $\frac{o\dot{z}}{m}$ will express the contempaneous increments of x, y, and z respectively. Now when x is become $x+\frac{o\dot{x}}{m}$, y is become $y+\frac{o\dot{y}}{m}$ and z become $z+\frac{o\dot{z}}{m}$; the former equation will become

$$x^3 + \frac{3x^2o\dot{x}}{m} + \frac{3xo^2\dot{x}\dot{x}}{m^2} + \frac{o^3\dot{x}^3}{m^3} - xy^2 - \frac{o\dot{x}y^2}{m} - \frac{2xo\dot{y}y}{m} - \frac{2\dot{x}o^2\dot{y}y}{m^2} - \frac{xo^2\dot{y}\dot{y}}{m^2} - \frac{\dot{x}o^3\dot{y}\dot{y}}{m^3} + a^2z + \frac{a^2o\dot{z}}{m} - b^3 = 0.$$

Here, as the first of these equations exhibits the relation between the three quantities x, y, z, as far as the same can be expressed by a single equation; so this second equation, with the assistance of the first, will express the relation between the augments of these quantities. But the first of these equations may be taken out of the latter; whence will arise this third equation

$$\frac{3x^2o\dot{x}}{m} + \frac{3xo^2\dot{x}\dot{x}}{m^2} + \frac{o^3\dot{x}^3}{m^3} - \frac{o\dot{x}y^2}{m} - \frac{2xo\dot{y}y}{m} - \frac{2\dot{x}o^2\dot{y}y}{m^2} - \frac{xo^2\dot{y}\dot{y}}{m^2} - \frac{\dot{x}o^3\dot{y}\dot{y}}{m^3} + \frac{a^2o\dot{z}}{m} = 0;$$

which also expresses the relation between the several increments; and likewise if o be a given quantity, this equation will equally express the relation between the velocities, wherewith these several increments are generated respectively by a uniform motion. And this equation being divided by o will be reduced to more simple terms, and yet will equally express the relation of these velocities; and then the equation will become

$$\frac{3x^2\dot{x}}{m} + \frac{3xo\dot{x}\dot{x}}{m^2} + \frac{o^2\dot{x}^3}{m^3} - \frac{\dot{x}y^2}{m} - \frac{2x\dot{y}y}{m} - \frac{2\dot{x}o\dot{y}y}{m^2} - \frac{xo\dot{y}\dot{y}}{m^2} - \frac{\dot{x}o^2\dot{y}\dot{y}}{m^3} + \frac{a^2\dot{z}}{m} = 0.$$

Now let us form an equation out of the terms of this, from which the quantity o is absent. This equation will be

$$\frac{3x^2\dot{x}}{m} - \frac{\dot{x}y^2}{m} - \frac{2x\dot{y}y}{m} + \frac{a^2\dot{z}}{m} = 0;$$

and this equation multiplied by m becomes

$$3x^2\dot{x} - \dot{x}y^2 - 2x\dot{y}y + a^2\dot{z} = 0.$$

It is evident, that this equation does not express the relation of the forementioned velocities; yet by the diminution of o this equation may come within any degree of expressing that relation. Therefore, by what has been so often inculcated, this equation will express the ultimate relation of these velocities. But the fluxions of the quantities x, y, z are the ultimate magnitudes of these velocities; so that the ultimate relation of these velocities is the relation of the fluxions of these quantities. Consequently this last equation represents the relation of the fluxions of the quantities x, y, z.

[150.] It is now presumed, we have removed all difficulty from the demonstrations, which Sir Isaac Newton has himself given, of his rules for finding fluxions.

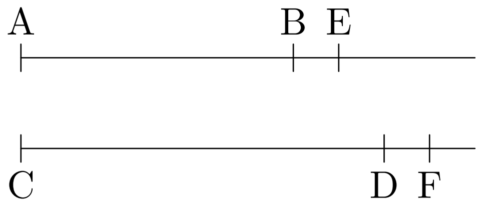
[151.] In the beginning of this discourse we have endeavoured at such a description of fluxions, as might not fail of giving a distinct and clear conception of them. We then confirmed the fundamental rules for comparing fluxions together by demonstrations of the most formal and unexceptionable kind. And now having justified Sir Isaac Newton's own demonstrations, we have not only shewn, that his doctrine of fluxions is an unerring guide in the solution of geometrical problems, but also that he himself had fully proved the certainty of this method. For accomplishing this last part of our undertaking it was necessary to explain at large another method of reasoning established by him, no less worthy consideration; since as the first inabled him to investigate the geometrical problems, whereby he was conducted in those remote searches into nature, which have been the subject of universal admiration, so to the latter method is owing the surprizing brevity, wherewith he has demonstrated those discoveries.

CONCLUSION.

[152.] Thus we have at length finished the whole of our design, and shall therefore put a period to this discourse with the explanation of the term momentum frequently used by Sir Isaac Newton, though we have yet had no occasion to mention it.

[153.] And in this I shall be the more particular, because Sir Isaac Newton's definition of momenta, That they are momentaneous increments or decrements of varying quantities, may possibly be thought obscure. Therefore I shall give a fuller delineation of them, and such a one, as shall agree to the general sense of his description, and never fail to make the use of this term, in every proposition, where it occurs, clearly to be understood.

[154.] In determining the ultimate ratios between the contemporaneous differences of quantities, it is often previously required to consider each of these differences apart, in order to discover, how much of those differences is necessary for expressing that ultimate ratio. In this case Sir Isaac Newton distinguishes, by the name of momentum, so much of any difference, as constitutes the term used in expressing this ultimate ratio.



[155.] Thus in [§ 147], where B E is $= o$, and D F equal to

$$nx^{n-1}o + \frac{n \times \overline{n-1}}{2} \times x^{n-2}o^2 + \frac{n \times \overline{n-1} \times \overline{n-2}}{6} \times x^{n-3}o^3 + \&c.$$

the ultimate ratio of D F to B E being the ratio of $nx^{n-1}o$ to o, such a part only of D F as is denoted by $nx^{n-1}o$, without the addition of any of the following terms of the series, constitutes the whole of the momentum of the line C D; but the momentum of A B is the same as the whole difference B E, or o.

[156.] In like manner, if A and B denote varying quantities, and their contemporaneous increments be represented by a and b; the rectangle under any given line M and a is the contemporaneous increment of the rectangle under M and A, and $\mathrm{A} \times b + \mathrm{B} \times a + a \times b$ is the like increment of the rectangle under A, B. And here the whole increment $\mathrm{M} \times a$ represents the momentum of the rectangle under M, A; but $\mathrm{A} \times b + \mathrm{B} \times a$ only, and not the whole increment $\mathrm{A} \times b + \mathrm{B} \times a + a \times b$, is called the momentum of the rectangle under A, B; because so much only of this latter increment is required for determining the ultimate ratio of the increment of $M \times A$ to the increment of $A \times B$, this ratio being the same with the ultimate ratio of $M \times a$ to $A \times b + B \times a$; for the ultimate ratio of $A \times b + B \times a$ to $A \times b + B \times a + a \times b$ is the ratio of equality. Consequently the ultimate ratio of $M \times a$ to $A \times b + B \times a$ differs not from the ultimate ratio of $M \times a$ to $A \times b + B \times a + a \times b$.

[157.] These momenta equally relate to the decrements of quantities, as to their increments, and the ultimate ratio of increments, and of decrements at the same place is the same; therefore the momentum of any quantity may be determined, either by considering the increment, or the decrement of that quantity, or even by considering both together. And in determining the momentum of the rectangle under A and B Sir Isaac Newton has taken the last of these methods; because by this means the superfluous rectangle is sooner disengaged from the demonstration.

[158.] Here it must always be remembred, that the only use, which ought ever to be made of the momenta, is to compare them one with another, and for no other purpose than to determine the ultimate or prime proportion between the several increments or decrements, from whence they are deduced*. Herein the method of prime and ultimate ratios essentially differs from that of indivisibles; for in that method these momenta are considered absolutely as parts, whereof their respective quantities are actually composed. But though these momenta have no final magnitude, which would be necessary to make them parts capable of compounding a whole by accumulation; yet their ultimate ratios are as truly assignable as the ratios between any quantities whatever. Therefore none of the objections made against the doctrine of indivisibles are of the least weight against this method: but while we attend carefully to the observation here laid down, we shall be as secure against error, and the mind will receive as full satisfaction, as in any the most unexceptionable demonstration of Euclide.

[159.] We shall make no apology for the length of this discourse: for as we can scarce suspect, after what has been above written, that our readers will be at any loss to remove of themselves, whatever little difficulties may have arisen in this subject from the brevity of Sir Isaac Newton's expressions; so our time cannot be thought misemployed, if we shall at all have contributed, by a more diffusive phrase, to the easier understanding these extensive, and celebrated inventions.

FINIS.

* Neque spectatur magnitudo momentorum, set prima nascentium proportio. Newt. Princ. Phil. Lib. II. Lem. 2.