INTRODUCTION TO THE

QUADRATURE OF CURVES

By

Isaac Newton

Translated into English by

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Edited by David R. Wilkins

2002

NOTE ON THE TEXT

This translation of Isaac Newton's *Introductio ad Quadraturam Curvarum* appeared (with a translation of the following treatise *De Quadratura Curvarum* in the second volume of the *Lexicon Technicum* by John Harris, published in 1710, from whence the present text is taken.

Commas have been deleted before the three occurrences of '*&c.*' in the paragraph on the fluxion of x^n. Also in that paragraph, the expression '$\frac{nn-n}{2}\,ox^{n-2}$' was printed as '$\frac{nn-n}{2}\,oox^{n-2}$' but this has been corrected in accordance with both the sense of the passage and Newton's Latin text as it appears in the collection of mathematical tracts of Isaac Newton, *Analysis per quantitatum series, fluxiones, ac differentias: cum enumeratione linearum tertii ordinis*, edited by William Jones (London, 1711).

David R. Wilkins

Dublin, June 2002

A reference to 'Angle $b\,P\,C$' (4th line from bottom, p.2) has been corrected to read 'Angle $b\,B\,C$' in accordance with both the sense of the passage and Newton's Latin text. Also, on p.3, line 7, 'Points b and E' has been corrected to read 'Points b and e', and, on p.3, line 8, '$B\,C$ to $E\,e$ as $A\,b \times P\,B$, to $A\,e \times P\,E$' has been corrected to read $B\,C$ to $E\,e$ as $P\,B$ to $P\,E$', for the same reason.

Dublin, February 2014

QUADRATURE *of Curves*

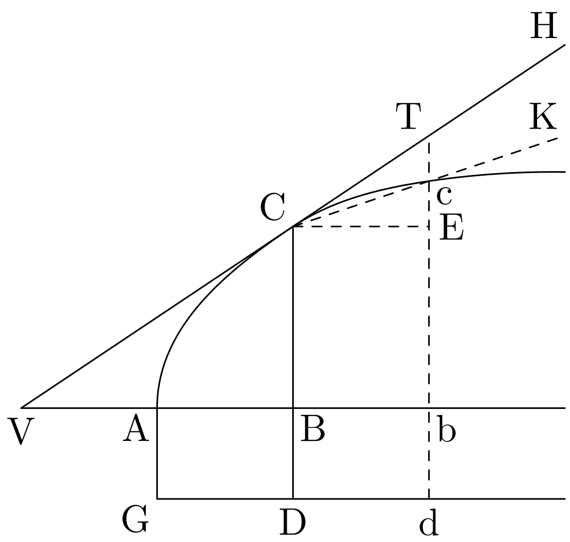
by Sir *Is. Newton.*

I don't here consider Mathematical Quantities as composed of Parts *extreamly small*, but as *generated by a continual motion.* Lines are described, and by describing are generated, not by any apposition of Parts, but by a continual motion of Points. Surfaces are generated by the motion of Lines, Solids by the motion of Surfaces, Angles by the Rotation of their Legs, Time by a continual flux, and so in the rest. These *Geneses* are founded upon Nature, and are every Day seen in the motion of Bodies.

And after this manner the Ancients by carrying moveable right Lines along immoveable ones in a Normal Position or Situation, have taught us the Geneses of Rectangles.

Therefore considering that Quantities, encreasing in equal times, and generated by this encreasing, are greater or less, according as their Velocity by which they encrease, and are generated, is greater or less; I endeavoured after a Method of determining the Quantities from the Velocities of their Motions or Increments, by which they are generated; and by calling the Velocities of the Motions, or of the Augments, by the Name of *Fluxions*, and the generated Quantities *Fluents*, I (in the years 1665 and 1666) did, by degrees, light upon the Method of *Fluxions*, which I here make use of in the *Quadrature of Curves.*

Fluxions are very nearly as the Augments of the Fluents, generated in equal, but infinitely small parts of Time; and to speak exactly, are in the *Prime Ratio* of the nascent Augments: but they may be expounded by any Lines that are proportional to 'em. As if the *Areas* ABC, $ABDG$ be described by the Ordinates BC, BD, moving with an uniform motion along the Base AB, the Fluxions of these *Areas* will be to one another as the describent Ordinates BC and BD, and may be expounded by those Ordinates; for those Ordinates are in the same Proportion as the Nascent Augments of the Areas.



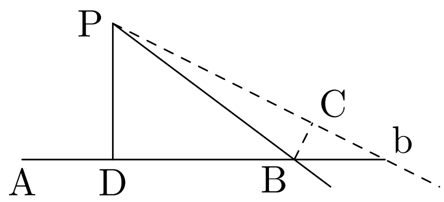
Let the Ordinate BC move out of its place BC into any new one bc: Compleat the Parallelogram $BCEb$, and let the Right Line VTH be drawn which may touch the Curve C and meet bc and BA produced in T and V; and then the just now generated Augments of the Abscissa AB, the Ordinate BC, and the Curve Line ACc, will be Bb, Ec and Cc; and the Side of the Triangle CET, are in the *Prime Ratio* of these Nascent Augments, and therefore the Fluxions of AB, BC and AC are as the Sides CE, ET and CT of the Triangle CET, and may be expounded by those Sides, or which is much at one, by the Sides of the Triangle VBC similar to it.

'Tis the same thing if the Fluxions be taken in the *ultimate Ratio* of the Evanescent Parts. Draw the Right Line Cc, and produce the same to K. Let the Ordinate bc return into its former place BC, and the points C and c coming together, the Right Line CK co-incides with the Tangent CH, and the Evanescent Triangle CEc in its ultimate form becomes similar to the Triangle CET, and its Evanescent Sides CE, Ec and Cc will be ultimately to one another as are CE, ET and CT the Sides of the other Triangle CET, and therefore the Fluxions of the Lines AB, BC and AC are in the same *Ratio*. If the Points C and c be at any small distance from one another, then will CK be at a small distance from the Tangent CH. As soon as the Right Line CK coincides with the Tangent CH, and the ultimate Ratio's of the Lines CE, Ec and Cd be found, the Points C and c ought to come together and exactly to coincide. For errours, tho' never so small, are not to be neglected in Mathematicks.

By the same way of arguing, if a Circle described on the Centre B with the Radius BC, be drawn with an uniform motion along the Abscissa AB, and at Right Angles to it, the Fluxion of the generated Solid ABC will be as the generating Circle, and the Fluxion of its Surface will be as the Perimeter of that Circle and the Fluxion of the Curve Line AC conjointly. For in what time the Solid ABC is generated, by drawing the Circle along the Abscissa AB, in the same time its Surface is generated by drawing the Perimeter of that Circle along the Curve AC.

Of this Method take the following Examples.

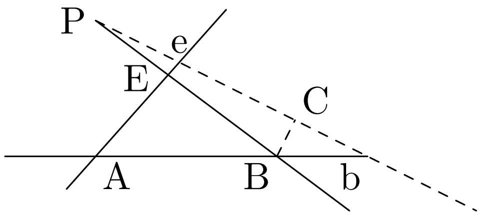
Let the Right Line PB *revolving about the given Pole* P *cut the Right Line* AB *given in Position; the Proportions of the Fluxions of the Right Line* AB *and* PB *is required.*



Let the Right Line PB go out of its place PB into a new one Pb: in the Line Pb take PC equal to PB and draw PD to AB so that the Angle bPD may be equal to the Angle bBC; and then from the Similarity of the Triangles bBC, bPD, the Augment Bb, will be to the Augment Cb as Pb is to Db.

Now let Pb return into its former place PB, that those Augments may vanish, and the ultimate Ratio of the Evanescent Augments, that is, the ultimate Ratio of Pb to Db will be the same as that of PB to DB, the Angle being right; and therefore the Fluxion of AB is to the Fluxion of PB in this Ratio.

Let the Right Line P B *revolving about the given Pole* P *cut* A B *and* A E *two other Right Lines given in Position in* B *and* E*; 'tis required to find the Proportion of the Fluxions of those Right Lines* A B *and* A E.



Let the revolving Line PB move out of its place PB into a new one Pb, cutting AB, AE into the Points b and e, and draw BC parallel to AE, meeting Pb in C; then Bb will be to BC as Ab is to Ae; and BC to Ee as PB to PE. Now let the Right Line Pb return into its former place PB, and the Evanescent Augment B b will be to the Evanescent Augment Ee as $AB \times PB$ is to $AE \times PE$, and therefore in this Ratio is the Fluxion of the Right Line AE.

Hence if the revolving Right Line PB cut any Curve Lines given in position in the Points B and E, and the moveable Right Lines AB, AE touch those Curves in B and E, the Points of Section; the Fluxion of the Curve which the Right Line AB touches, will be to the Fluxion of the Curve which the Right Line AE touches, as $AB \times PB$ is to $AE \times PE$. The same thing will happen if the Right Line PB always touch any Curve given in Position in the moveable Point P.

Let the Quantity x flow uniformly, and let the Fluxion of x^n be to be found. In the same time that the Quantity x by flowing becomes $x + o$, the Quantity x^n will become $\overline{x+o}|^n$, that is, by the Method of Infinite Series's

$$x^n + nox^{n-1} + \frac{nn-n}{2} oox^{n-2} + \&c.$$

and the Augments

$$o \quad \text{and} \quad nox^{n-1} + \frac{nn-n}{2} oox^{n-2} + \&c.$$

are to one another as

$$1 \quad \text{and} \quad nx^{n-1} + \frac{nn-n}{2} ox^{n-2} + \&c.$$

Now let those Augments vanish and their ultimate Ratio will be the Ratio of 1 to nx^{n-1}; and therefore the Fluxion of the Quantity x is to the Fluxion of the Quantity x^n as 1 to nx^{n-1}.

By like ways of arguing, and by the method of Prime and Ultimate Ratio's, may be gathered the Fluxions of Lines, whether Right or Crooked in all cases whatsoever, as also the Fluxions of Surfaces, Angles and other Quantities. In Finite Quantities so to frame a Calculus, and thus to investigate the Prime and Ultimate Ratio's of Nascent or Evanescent

Finite Quantities, is agreeable to the Geometry of the Ancients; and I was willing to shew, that in the Method of Fluxions there's no need of introducing Figures infinitely small into Geometry. For this Analysis may be performed in any Figures whatsoever, whether finite or infinitely small, so they are but imagined to be similar to the Evanescent Figures; as also in Figures which may be reckoned as infinitely small, if you do but proceed cautiously.

From the Fluxions to find the Fluents is the more difficult *Problem*, and the 1st step of the Solution of it is equivalent to the *Quadrature of Curves*; concerning which I have formerly written the following Tract.