THE MATHEMATICAL PRINCIPLES OF NATURAL PHILOSOPHY

(BOOK 2, LEMMA 2)

By

Isaac Newton

Translated into English by

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NOTE ON THE TEXT

Lemma II in Book II of Isaac Newton's *Philosophiæ Naturalis Principia Mathematica* is reproduced here, translated into English by Andrew Motte. Motte's translation of Newton's *Principia*, entitled *The Mathematical Principles of Natural Philosophy* was first published in 1729.

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Lemma II.

The moment of any Genitum *is equal to the moments of each of the generating sides drawn into the indices of the powers of those sides, and into their coefficients continually.*

I call any quantity a *Genitum*, which is not made by addition or subduction of divers parts, but is generated or produced in arithmetic by the multiplication, division, or extraction of the root of any terms whatsoever; in geometry by the invention of contents and sides, or of the extreams and means of proportionals. Quantities of this kind are products, quotients, roots, rectangles, squares, cubes, square and cubic sides and the like. These quantities I here consider as variable and indetermined, and increasing or decreasing as it were by a perpetual motion or flux; and I understand their momentaneous increments or decrements by the name of Moments; so that the increments may be esteem'd as added, or affirmative moments; and the decrements as subducted, or negative ones. But take care not to look upon finite particles as such. Finite particles are not moments, but the very quantities generated by the moments. We are to conceive them as the just nascent principles of finite magnitudes. Nor do we in this Lemma regard the magnitude of the moments, but their first proportion as nascent. It will be the same thing, if, instead of moments, we use either the Velocities of the increments and decrements (which may also be called the motions, mutations, and fluxions of quantities) or any finite quantities proportional to those velocities. The coefficient of any generating side is the quantity which arises by applying the Genitum to that side.

Wherefore the sense of the Lemma is, that if the moments of any quantities A, B, C, &c. increasing or decreasing by a perpetual flux, or the velocities of the mutations which are proportional to them, be called a, b, c, &c. the moment or mutation of the generated rectangle AB will be $aB + bA$; the moment of the generated content ABC will be $aBC + bAC + cAB$: and the moments of the generated powers A^2, A^3, A^4, $A^{\frac{1}{2}}$, $A^{\frac{3}{2}}$, $A^{\frac{1}{3}}$, $A^{\frac{2}{3}}$, A^{-1}, A^{-2}, $A^{-\frac{1}{2}}$ will be $2aA$, $3aA^2$, $4aA^3$, $\frac{1}{2}aA^{-\frac{1}{2}}$, $\frac{3}{2}aA^{\frac{1}{2}}$, $\frac{1}{3}aA^{-\frac{2}{3}}$, $\frac{2}{3}aA^{-\frac{1}{3}}$, $-aA^{-2}$, $-2aA^{-3}$, $-\frac{1}{2}aA^{-\frac{3}{2}}$ respectively. And in general, that the moment of any power $A^{\frac{n}{m}}$ will be $\dfrac{n}{m}aA^{\frac{n-m}{m}}$. Also that the moment of the generated quantity A^2B will be $2aAB + bA^2$; the moment of the generated quantity $A^3B^4C^2$ will be $3aA^2B^4C^2 + 4bA^3B^3C^2 + 2cA^3B^4C$; and the moment of the generated quantity $\dfrac{A^3}{B^2}$ or A^3B^{-2} will be $3aA^2B^{-2} - 2bA^3B^{-3}$; and so on. The Lemma is thus demonstrated.

Case 1. Any rectangle as AB augmented by a perpetual flux, when, as yet, there wanted of the sides A and B half their moments $\frac{1}{2}a$ and $\frac{1}{2}b$, was $A - \frac{1}{2}a$ into $B - \frac{1}{2}b$, or $AB - \frac{1}{2}aB - \frac{1}{2}bA + \frac{1}{4}ab$; but as soon as the sides A and B are augmented by the other half moments; the rectangle becomes $A + \frac{1}{2}a$ into $B + \frac{1}{2}b$, or $AB + \frac{1}{2}aB + \frac{1}{2}bA + \frac{1}{4}ab$; From this rectangle subduct the former rectangle, and there will remain the excess $aB + bA$. Therefore with the whole increments a and b of the sides, the increment $aB + bA$ of the rectangle is generated. *Q.E.D.*

Case 2. Suppose AB always equal to G, and then the moment of the content ABC or GC (by Case 1.) will be $gC + cG$, that is, (putting AB and $aB + bA$ for G and g) $aBC + bAC + cAB$. And the reasoning is the same for contents, under never so many sides. *Q.E.D.*

Case 3. Suppose the sides A, B, and C, to be always equal among themselves; and the moment $aB + bA$, of A^2, that is, of the rectangle AB, will be $2aA$; and the moment $aBC + bAC + cAB$ of A^3, that is, of the content ABC, will be $3aA^2$. And by the same reasoning the moment of any power A^n is naA^{n-1}. *Q.E.D.*

Case 4. Therefore since $\frac{1}{A}$ into A is 1, the moment of $\frac{1}{A}$ drawn into A, together with $\frac{1}{A}$ drawn into a, will be the moment of 1, that is, nothing. Therefore the moment of $\frac{1}{A}$ or of A^{-1} is $\frac{-a}{A^2}$. And generally, since $\frac{1}{A^n}$ into A^n is 1, the moment of $\frac{1}{A^n}$ drawn into A^n together with $\frac{1}{A^n}$ into naA^{n-1} will be nothing. And therefore the moment of $\frac{1}{A^n}$ or A^{-n} will be $\frac{-na}{A^{n+1}}$. *Q.E.D.*

Case 5. And since $A^{\frac{1}{2}}$ into $A^{\frac{1}{2}}$ is A, the moment of $A^{\frac{1}{2}}$ drawn into $2A^{\frac{1}{2}}$ will be a, (by Case 3:) and therefore the moment of $A^{\frac{1}{2}}$ will be $\frac{a}{2A^{\frac{1}{2}}}$ or $\frac{1}{2}aA^{-\frac{1}{2}}$. And generally putting $A^{\frac{m}{n}}$ equal to B, then A^m will be equal to B^n, and therefore maA^{m-1} equal to nbB^{n-1}, and maA^{-1} equal to nbB^{-1} or $nbA^{-\frac{m}{n}}$; and therefore $\frac{m}{n}aA^{\frac{m-n}{n}}$ is equal to b, that is, equal to the moment of $A^{\frac{m}{n}}$. *Q.E.D.*

Case 6. Therefore the moment of any generated quantity A^mB^n is the moment of A^m drawn into B^n, together with the moment of B^n drawn into A^m, that is, $maA^{m-1}B^n + nbB^{n-1}A^m$; and that whether the indices m and n of the powers be whole numbers or fractions, affirmative or negative. And the reasoning is the same for contents under more powers. *Q.E.D.*

Cor. 1 Hence in quantities continually proportional, if one term is given, the moments of the rest of the terms will be as the same terms multiplied by the number of intervals between them and the given term. Let A, B, C, D, E, F, be continually proportional; then if the term C is given, the moments of the rest of the terms will be among themselves, as $-2A$, $-B$, D, $2E$, $3F$.

Cor. 2 And if in four proportionals the two means are given, the moments of the extremes will be as those extremes. The same is to be understood of the sides of any given rectangle.

Cor. 3. And if the sum or difference of two squares is given, the moments of the sides will be reciprocally as the sides.

Scholium.

In a letter of mine to Mr. *J. Collins*, dated *December* 10. 1672 having described a method of Tangents, which I suspected to be the same with *Slusius*'s method, which at that time was not made publick; I subjoined these words; *This is one particular, or rather a corollary, of a general method, which extends itself, without any troublesome calculation, not only to the drawing of Tangents to any Curve lines, whether Geometrical or Mechanical, or any* *how respecting right lines or other Curves, but also to the resolving other abstruser kinds of Problems about the crookedness, areas, lengths, centres of gravity of Curves, &c. nor is it (as* Hudden's *method* de Maximis & Minimis*) limited to equations which are free from surd quantities. This method I have interwoved with that other of working in equations, by reducing them to infinite series.* So far that letter. And these last words relate to a Treatise I composed on that subject in the year 1671. The foundation of the general method is contained in the preceding Lemma.