THE MATHEMATICAL PRINCIPLES OF NATURAL PHILOSOPHY

(BOOK 1, SECTION 1)

By

Isaac Newton

Translated into English by

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2002

NOTE ON THE TEXT

Section I in Book I of Isaac Newton's *Philosophiæ Naturalis Principia Mathematica* is reproduced here, translated into English by Andrew Motte. Motte's translation of Newton's *Principia*, entitled *The Mathematical Principles of Natural Philosophy* was first published in 1729.

David R. Wilkins
Dublin, June 2002

SECTION I.

Of the method of first and last ratio's of quantities, by the help whereof we demonstrate the propositions that follow.

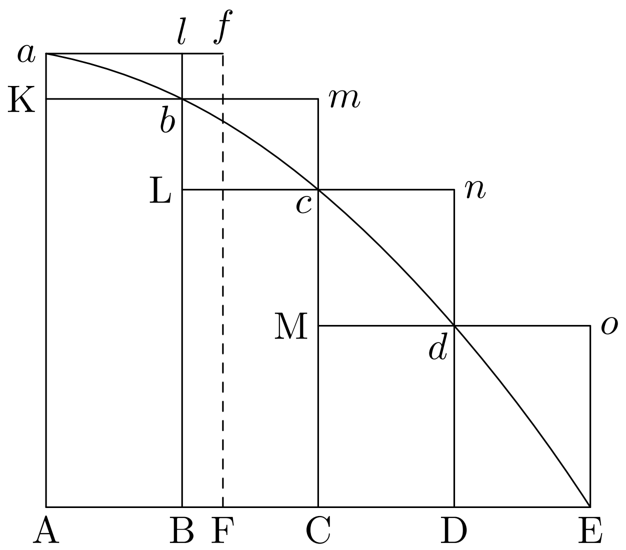
Lemma I.

Quantities, and the ratio's of quantities, which in any finite time converge continually to equality, and before the end of that time approach nearer the one to the other than by any given difference, become ultimately equal.

If you deny it; suppose them to be ultimately unequal, and let D be their ultimate difference. Therefore they cannot approach nearer to equality than by that given difference D; which is against the supposition.

Lemma II.

If in any figure AacE *terminated by the right lines* Aa, AE, *and the curve* acE, *there be inscrib'd any number of parallelograms* Ab, Bc, Cd, *&c. comprehended under equal bases* AB, BC, CD, *&c. and the sides* Bb, Cc, Dd, *&c. parallel to one side* Aa *of the figure; and the parallelograms* aKbl, bLcm, cMdn, *&c. are compleated. Then if the breadth of those parallelograms be suppos'd to be diminished, and their number to be augmented* in infinitum: *I say that the ultimate ratio's which the inscrib'd figure* AKbLcMdD, *the circumscribed figure* AalbmcndoE, *and the curvilinear figure* AabcdE, *will have to one another, are ratio's of equality.*



For the difference of the inscrib'd and circumscrib'd figures is the sum of the parallelograms $K\,l$, $L\,m$, $M\,n$, $D\,o$, that is, (from the equality of all their bases) the rectangle under one of their bases $K\,b$ and the sum of their altitudes $A\,a$, that is, the rectangle $A\,B\,l\,a$. But this rectangle, because its breadth $A\,B$ is suppos'd diminished *in infinitum*, becomes less than any given space. And therefore (By Lem. I.) the figures inscribed and circumscribed become ultimately equal one to the other; and much more will the intermediate curvilinear figure be ultimately equal to either. *Q.E.D.*

LEMMA III.

The same ultimate ratio's are also ratio's of equality, when the breadths, A B, B C, D C, *&c. of the parallelograms are unequal, and are all diminished* in infinitum.

For suppose $A\,F$ equal to the greatest breadth, and compleat the parallelogram F A a f. This parallelogram will be greater than the difference of the inscrib'd and circumscribed figures; but, because its breadth $A\,F$ is diminished *in infinitum*, it will become less than any given rectangle. *Q.E.D.*

COR. 1. Hence the ultimate sum of those evanescent parallelograms will in all parts coincide with the curvilinear figure.

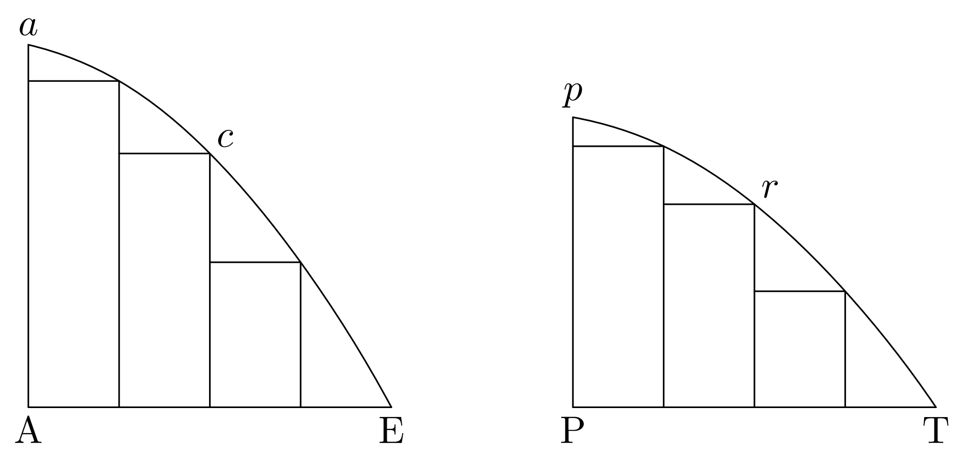
COR. 2. Much more will the rectilinear figure, comprehended under the chords of the evanescent arcs $a\,b$, $b\,c$, $c\,d$, &c. ultimately coincide with the curvilinear figure.

COR. 3. And also the circumscrib'd rectilinear figure comprehended under the tangents of the same arcs.

COR. 4. And therefore these ultimate figures (as to their perimeters $a\,c\,E$,) are not rectilinear, but curvilinear limits of rectilinear figures.

LEMMA IV.

If in two figures A a c E, P p r T *you inscribe (as before) two ranks of parallelograms, an equal number in each rank, and when their breadths are diminished* in infinitum, *the ultimate ratio's of the parallelograms in one figure to those in the other, each to each respectively, are the same; I say that those two figures* A a c E, P p r T, *are to one another in that same ratio.*



For as the parallelograms in the one are severally to the parallelograms in the other, so (by composition) is the sum of all in the one to the sum of all in the other; and so is the one figure to the other; because (by Lem. 3.) the former figure to the former sum, and the latter figure to the latter sum are both in the ratio of equality. *Q.E.D.*

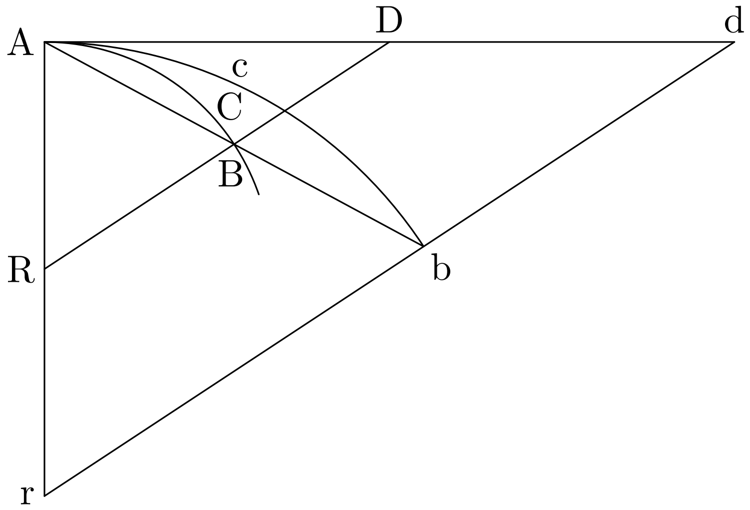
COR. Hence if two quantities of any kind are any how divided into an equal number of parts: and those parts, when their number is augmented and their magnitude diminished *in infinitum*, have a given ratio one to the other, the first to the first, the second to the second, and so on in order: the whole quantities will be one to the other in that same given ratio. For if, in the figures of this lemma, the parallelograms are taken one to the other in the ratio of the parts, the sum of the parts will always be as the sum of the parallelograms; and therefore supposing the number of the parallelograms and parts to be augmented, and their magnitudes diminished *in infinitum*, those sums will be in the ultimate ratio of the parallelogram in the one figure to the correspondent parallelogram in the other; that is, (by the supposition) in the ultimate ratio of any part of the one quantity to the correspondent part of the other.

LEMMA V.

In similar figures, all sorts of homologous sides, whether curvilinear or rectilinear, are proportional; and the area's are in the duplicate ratio of the homologous sides.

LEMMA VI.

If any arc A C B *given in position is subtended by its chord* A B*, and in any point* A *in the middle of the continued curvature, is touch'd by a right line* A D*, produced both ways; then if the points* A *and* B *approach one another and meet, I say the angle* B A D*, contained between the chord and the tangent, will be diminished in infinitum, and ultimately will vanish.*

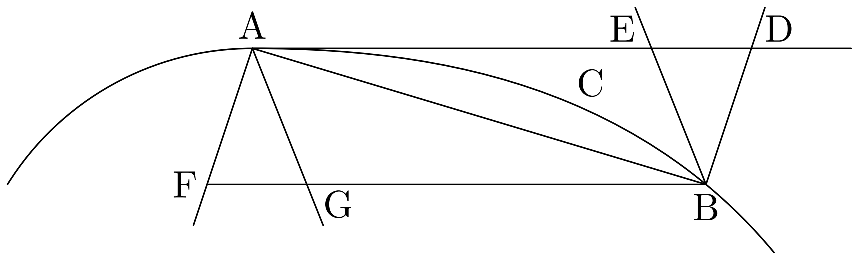


For if that angle does not vanish, the arc ACB will contain with the tangent AD an angle equal to a rectilinear angle; and therefore the curvature at the point A will not be continued, which is against the supposition.

Lemma VII.

The same things being supposed; I say, that the ultimate ratio of the arc, chord, and tangent, any one to any other, is the ratio of equality.

For while the point B approaches towards the point A, consider always AB and AD as produc'd to the remote points b and d, and parallel to the secant BD draw bd: and let the arc Acb be always similar to the arc ACB. Then supposing the points A and B to coincide, the angle dAb will vanish, by the preceding lemma; and therefore the right lines Ab, Ad (which are always finite) and the intermediate arc Acb will coincide, and become equal among themselves. Wherefore the right lines AB, AD, and the intermediate arc ACB (which are always proportional to the former) will vanish; and ultimately acquire the ratio of equality. *Q.E.D.*



Cor. 1. Whence if through B we draw BF parallel to the tangent, always cutting any right line AF passing through A in F; this line BF will be ultimately in the ratio of equality with the evanescent arc ACB; because, compleating the parallelogram $AFBD$, it is always in a ratio of equality with AD.

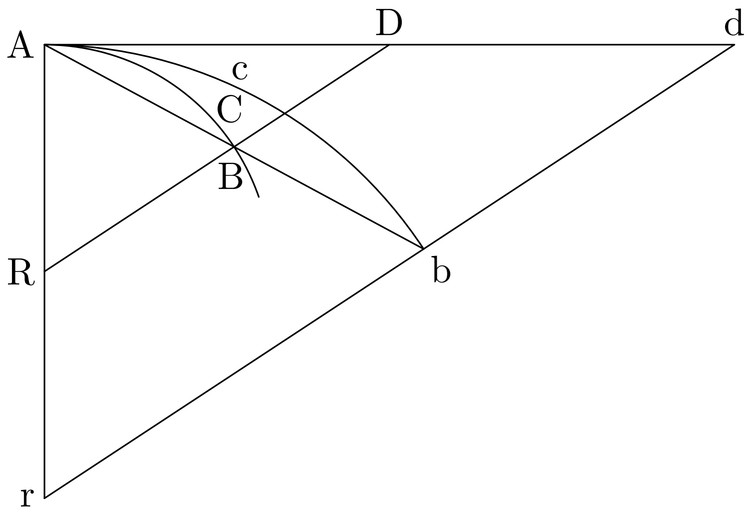
Cor. 2. And if through B and A more right lines are drawn as BE, BD, AF, AG cutting the tangent AD and its parallel BF; the ultimate ratio of all the abscissa's AD, AE, BF, BG, and of the chord and arc AB, any one to any other, will be the ratio of equality.

Cor. 3. And therefore in all our reasoning about ultimate ratio's, we may freely use any one of those lines for any other.

Lemma VIII.

If the right lines AR*,* BR*, with the arc* ACB*, the chord* AB*, and the tangent* AD*, constitute three triangles* RAB*,* RACB*,* RAD*, and the points* A *and* B *approach and meet: I say that the ultimate form of these evanescent triangles is that of similitude, and their ultimate ratio that of equality.*

For while the point B approaches towards the point A consider always AB, AD, AR, as produced to the remote points b, d, and r, and rbd as drawn parallel to RD, and let the arc Acb be always similar to the arc ACB. Then supposing the points A and B to coincide, the angle bAd will vanish; and therefore the three triangles rAb, $rAcb$, rAd, (which are always finite) will coincide, and on that account become both similar and equal. And therefore the

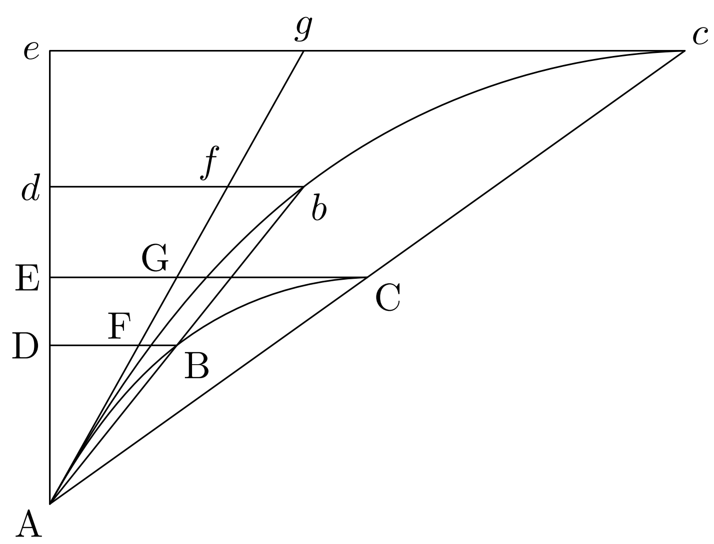


triangles RAB, $RACB$, RAD, which are always similar and proportional to these. will ultimately become both similar and equal among themselves. *Q.E.D.*

COR. And hence in all our reasonings about ultimate ratio's, we may indifferently use any one of those triangles for any other.

LEMMA IX.

If a right line AE*, and a curve line* ABC*, both given by position, cut each other in a given angle* A*; and to that right line, in another given angle,* BD, CE*, are ordinately applied, meeting the curve in* B, C*; and the points* B *and* C *together, approach towards, and meet in, the point* A*: I say that the area's of the triangles* ABD*,* ACE*, will ultimately be one to the other in the duplicate ratio of the sides.*



For while the points B, C approach towards the point A, suppose always AD to be produced to the remote points d and e, so as Ad, Ae may be proportional to AD, AE; and the ordinates db, ec, to be drawn parallel to the ordinates DB and EC, and meeting AB

and AC produced in b and c. Let the curve Abc be similar to the curve ABC, and draw the right line Ag so as to touch both curves in A, and cut the ordinates DB, EC, db, ec, in F, G, f, g. Then supposing the length Ae to remain the same, let the points B and C meet in the point A; and the angle cAg vanishing, the curvilinear areas Abd, Ace will coincide with the rectilinear areas Afd, Age; and therefore (by Lem. 5) will be one to the other in the duplicate ratio of the sides Ad, Ae. But the areas ABD, ACE are always proportional to these areas, and so the sides AD, AE are to these sides. And therefore the areas ABD, ACE are ultimately one to the other in the duplicate ratio of the sides AD, AE. *Q.E.D.*

Lemma X.

The spaces which a body describes by any finite force urging it, whether that force is determined and immutable, or is continually augmented or continually diminished, are in the very beginning of the motion one to the other in the duplicate ratio of the times.

Let the times be represented by the lines AD, AE, and the velocities generated in those times by the ordinates DB, EC. The spaces described with these velocities will be as the areas ABD, ACE, described by those ordinates, that is, at the very beginning of the motion (by Lem. 9) in the duplicate ratio of the times AD, AE. *Q.E.D.*

Cor. 1. And hence one may easily infer, that the errors of bodies describing similar parts of similar figures in proportional times, are nearly in the duplicate ratio of the times in which they are generated; if so be these errors are generated by any equal forces similarly applied to the bodies, and measur'd by the distances of the bodies from those places of the similar figures, at which, without the action of those forces, the bodies would have arrived in those proportional times.

Cor. 2. But the errors that are generated by proportional forces similarly applied to the bodies at similar parts of the similar figures, are as the forces and the squares of the times conjunctly.

Cor. 3. The same thing is to be understood of any spaces whatsoever described by bodies urged with different forces. All which, in the very beginning of the motion, are as the forces and the squares of the times conjunctly.

Cor. 4. And therefore the forces are as the spaces described described in the very beginning of the motion directly, and the squares of the times inversly.

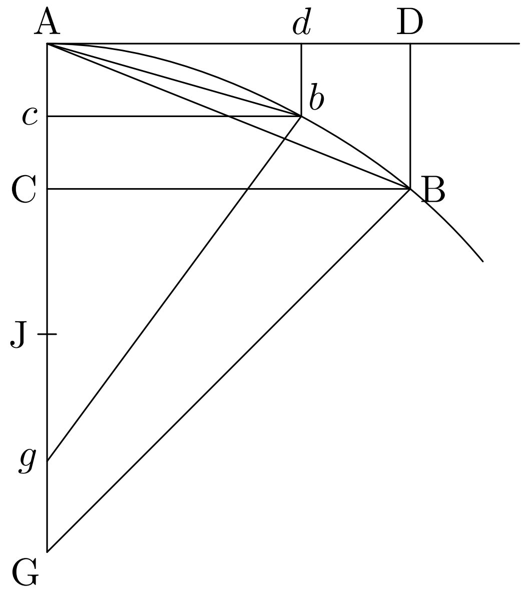
Cor. 5. And the squares of the times are as the spaces describ'd directly and the forces inversly.

Scholium.

If in comparing indetermined quantities of different sorts one with another, any one is said to be as any other directly or inversly: the meaning is, that the former is augmented or diminished in the same ratio with the latter, or with its reciprocal. And if any one is said to be as any other two or more directly or inversly: the meaning is, that the first is augmented or diminished in the ratio compounded of the ratio's in which the others, or the reciprocals of the others, are augmented or diminished. As if A is said to be as B directly and C directly and D inversly: the meaning is, that A is augmented or diminished in the same ratio with $B \times C \times \frac{1}{D}$, that is to say, that A and $\frac{BC}{D}$ are one to the other in a given ratio.

Lemma XI.

The evanescent subtense of the angle of contact, in all curves, which at the point of contact have a finite curvature, is ultimately in the duplicate ratio of the subtense of the conterminate arc.



Case 1. Let $A\,B$ be that arc, $A\,D$ its tangent, $B\,D$ the subtense of the angle of contact perpendicular on the tangent, $A\,B$ the subtense of the arc. Draw $B\,G$ perpendicular to the subtense $A\,B$, and $A\,G$ to the tangent $A\,D$, meeting in G; then let the points D, B and G, approach to the points d, b and g, and suppose J to be the ultimate intersection of the lines $B\,G$, $A\,G$, when the points D, B have come to A. It is evident that the distance $G\,J$ may be less than any assignable. But (from the nature of the circles passing through the points A, B, G; A, b, g) $A\,B^2 = A\,G \times B\,D$, and $A\,b^2 = A\,g \times b\,d$; and therefore the ratio of $A\,B^2$ to $A\,b^2$ is compounded of the ratio's of $A\,G$ to $A\,g$ and of $B\,D$ to $b\,d$. But because $G\,J$ may be assum'd of less length than any assignable, the ratio of $A\,G$ to $A\,g$ may be such as to differ from the ratio of equality by less than any assignable difference; and therefore the ratio of $A\,B^2$ to $A\,b^2$ may be such as to differ from the ratio of $B\,D$ to $b\,d$ by less than any assignable difference. Therefore, by Lem. 1. the ultimate ratio of $A\,B^2$ to $A\,b^2$ is the same with the ultimate ratio of $B\,D$ to $b\,d$. *Q.E.D.*

Case 2. Now let $B\,D$ be inclined to $A\,D$ in any given angle, and the ultimate ratio of $B\,D$ to $b\,d$ will always be the same as before, and therefore the same with the ratio of $A\,B^2$ to $A\,b^2$. *Q.E.D.*

Case 3. And if we suppose the angle D not to be given, but that the right line $B\,D$ converges to a given point, or is determined by any other condition whatever; nevertheless, the angles D, d, being determined by the same law, will always draw nearer to equality, and approach nearer to each other than by any assigned difference, and therefore, by Lem. 1, will at last be equal, and therefore the lines $B\,D$, $b\,d$ are in the same ratio to each other as before. *Q.E.D.*

COR. 1. Therefore since the tangents AD, Ad, the arcs AB, Ab, and their sines BC, bc, become ultimately equal to the chords AB, Ab; their squares will ultimately become as the subtenses BD, bd.

COR. 2. Their squares are also ultimately as the versed sines of the arcs, bisecting the chords, and converging to a given point. For those versed sines are as the subtenses BD, bd.

COR. 3. And therefore the versed sine is in the duplicate ratio of the time in which a body will describe an arc with a given velocity.

COR. 4. The rectilinear triangles ADB, Adb are ultimately in the triplicate ratio of the sides AD, Ad, and in a sesquiplicate ratio of the sides DB, db; as being in the ratio compounded of the sides AD to DB, and of Ad to db. So also the triangles ABC, Abc are ultimately in the triplicate ratio of the sides BC, bc. What I call the sesquiplicate ratio is the subduplicate of the triplicate, as been compounded of the simple and subduplicate ratio.

COR. 5. And because DB, db are ultimately parallel and in the duplicate ratio of the lines AD, Ad: the ultimate curvilinear areas ADB, Adb will be (by the nature of the parabola) two thirds of the rectilinear triangles ADB, Adb; and the segments AB, Ab will be one third of the same triangles. And thence those areas and those segments will be in the triplicate ratio as well of the tangents AD, Ad; as of the chords and arcs AB, Ab.

SCHOLIUM.

But we have all along supposed the angle of contact to be neither infinitely greater nor infinitely less, than the angles of contact made by circles and their tangents; that is, that the curvature at the point A is neither infinitely small nor infinitely great, or that the interval AJ is of a finite magnitude. For DB may be taken as AD^3: in which case no circle can be drawn through the point A, between the tangent AD and the curve AB, and therefore the angle of contact will be infinitely less than those of circles. And by a like reasoning if DB be made successively as AD^4, AD^5, AD^6, AD^7, &c. we shall have a series of angles of contact, proceeding *in infinitum*, wherein every succeeding term is infinitely less than the preceding. And if DB be made successively as AD^2, $AD^{\frac{3}{2}}$, $AD^{\frac{4}{3}}$, $AD^{\frac{5}{4}}$, $AD^{\frac{6}{5}}$, $AD^{\frac{7}{6}}$, &c. we shall have another infinite series of angles of contact, the first of which is of the same sort with those of circles, the second infinitely greater, and every succeeding one infinitely greater than the preceding. But between any two of these angles another series of intermediate angles of contact may be interposed proceeding both ways *in infinitum*, wherein every succeeding angle shall be infinitely greater, or infinitely less than the preceding. As if between the terms AD^2 and AD^3 there were interposed the series $AD^{\frac{13}{6}}$, $AD^{\frac{11}{5}}$, $AD^{\frac{9}{4}}$, $AD^{\frac{7}{3}}$, $AD^{\frac{5}{2}}$, $AD^{\frac{8}{3}}$, $AD^{\frac{11}{4}}$, $AD^{\frac{14}{5}}$, $AD^{\frac{17}{6}}$, &c. And again between any two angles of this series, a new series of intermediate angles may be interpolated, differing from one another by infinite intervals. Nor is nature confin'd to any bounds.

Those things which have been demonstrated of curve lines and the superficies which they comprehend, may be easily applied to the curve superficies and contents of solids. These lemmas are premised, to avoid the tediousness of deducing perplexed demonstrations *ad absurdum*, according to the method of the ancient geometers. For demonstrations are more contracted by the method of indivisibles: But because the hypothesis of indivisibles seems somewhat harsh, and therefore that method is reckoned less geometrical; I chose rather to reduce the demonstrations of the following propositions to the first and last sums and ratio's of nascent and evanescent quantities, that is, to the limits of those sums and ratio's; and so to premise, as short as I could, the demonstrations of those limits. For hereby the same thing is perform'd as by the method of indivisibles; and now those principles being demonstrated, we may use them with more safety. Therefore if hereafter, I should happen to consider quantities as made up of particles, or should use little curve lines for right ones; I would not be understood to mean indivisibles, but evanescent divisible quantities; not the sums and ratio's of determinate parts, but always the limits of sums and ratio's: and that the force of such demonstrations always depends on the method lay'd down in the foregoing lemma's.

Perhaps it may be objected, that there is no ultimate proportion of evanescent quantities; because the proportion, before the quantities have vanished, is not the ultimate, and when they are vanished, is none. But by the same argument it may be alledged, that a body arriving at a certain place, and there stopping, has no ultimate velocity; because the velocity, before the body comes to the place, is not its ultimate velocity; when it has arrived, is none. But the answer is easy; for by the ultimate velocity is meant that with which the body is moved, neither before it arrives at its last place and the motion ceases, nor after, but at the very instant it arrives; that is, that velocity with which the body arrives at its last place, and with which the motion ceases. An in like manner, by the ultimate ratio of evanescent quantities is to be understood the ratio of the quantities, not before they vanish, nor afterwards, but with which they vanish. In like manner the first ratio of nascent quantities is that with which they begin to be. And the first or last sum is that with which they begin and cease to be (or to be augmented or diminished.) There is a limit which the velocity at the end of the motion may attain, but not exceed. This is the ultimate velocity. And there is the like limit in all quantities and proportions that begin and cease to be. And since such limits are certain and definite, to determine the same is a problem strictly geometrical. But whatever is geometrical we may be allowed to use in determining and demonstrating any other thing that is likewise geometrical.

It may also be objected, that if the ultimate ratio's of evanescent quantities are given, their ultimate magnitudes will be also given: and so all quantities will consist of indivisibles, which is contrary to what *Euclid* has demonstrated concerning incommensurables, in the 10th book of his Elements. But this objection is founded on a false supposition. For those ultimate ratio's with which quantities vanish, are not truly the ratio's of ultimate quantities, but limits towards which the ratio's of quantities, decreasing without limit, do always converge; and to which they approach nearer than by any given difference, but never go beyond, nor in effect attain to, till the quantities are diminished *in infinitum*. This thing will appear more evident in quantities infinitely great. If two quantities, whose difference is given, be augmented *in infinitum*, the ultimate ratio of these quantities will be given, to wit, the ratio of equality; but it does not from thence follow, that the ultimate or greatest quantities themselves, whose ratio that is, will be given. Therefore if in what follows, for the sake of being more easily understood, I should happen to mention quantities as least, or evanescent, or ultimate; you are not to suppose that the quantities of any determinate magnitude are meant, but such as are conceiv'd to be always diminished without end.