Extract from

THIRD SUPPLEMENT TO AN ESSAY ON THE THEORY OF SYSTEMS OF RAYS

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This extract consists of the following sections of the paper:

- Introduction;
- Table of Contents;
- Articles 1–3 (properties of the characteristic function V and its derivatives);
- Articles 26–31 (connection with the undulatory theory of light, refraction in biaxal crystals, conical refraction).

Third Supplement to an Essay on the Theory of Systems of Rays. By WILLIAM R. HAMILTON, *A. B., M. R. I. A., M. R. Ast. Soc. London, M. G. Soc. Dublin, Hon. M. Soc. Arts for Scotland, Hon. M. Portsmouth Lit. and Phil. Soc., Member of the British Association for the Advancement of Science, Fellow of the American Academy of Arts and Sciences, Andrews' Professor of Astronomy in the University of Dublin, and Royal Astronomer of Ireland.*

Read January 23, 1832 and October 22, 1832.

INTRODUCTION

The present Supplement contains a system of general methods for the solution of Optical Problems, together with some general results, deduced from the fundamental formula and view of Optics, which have been proposed in my former memoirs. The copious analytical headings, prefixed to the several numbers, and collected in the Table of Contents, will sufficiently explain the plan of the present communication, and it is only necessary to say a few words here, respecting some of the principal results.

Of these the theory of external and internal conical refraction, deduced by my general methods from the principles of FRESNEL will probably be thought the least undeserving of attention. It is right, therefore, to state that this theory had been deduced, and was communicated to a general meeting of the Royal Irish Academy, not at the earlier, but at the later of the two dates prefixed to the present Supplement. After making this communication to the Academy, in October, 1832, I requested Professor LLOYD to examine the question experimentally, and to try whether he could perceive any such phenomena in biaxal crystals, as my theory of conical refraction had led me to expect. The experiments of Professor LLOYD, confirming my theoretical expectations, have been published by him in the numbers of the London and Edinburgh Philosophical Magazine, for the months of February and March, 1833; and they will be found with fuller details in the present Volume of the Irish Transactions.

I am informed that James MAC CULLAGH, Esq., F.T.C.D., who published in the last preceding Volume of these Transactions a series of elegant Geometrical Illustrations of FRESNEL'S theory, has, since he heard of the experiments of Professor LLOYD, employed his own geometrical methods to confirm my results respecting the existence of those conoidal cusps and circles of contact on FRESNEL'S wave, from which I had been led to the expectation of conical refraction. And on my lately mentioning to him that I had connected these cusps and circles on FRESNEL'S wave, with circles and cusps of the same kind on a certain other surface discovered by M. CAUCHY, by a general theory of reciprocal surfaces, which I stated last year at a general meeting of the Royal Irish Academy, Mr. MAC CULLAGH said that he had arrived independently at similar results, and put into my hands a paper on the subject, which I have not yet been able to examine, but which will (I hope) be soon presented to the Academy, and published in their Transactions.

I ought also to mention, that on my writing in last November to Professor AIRY, and communicating to him my results respecting the cusps and circles of FRESNEL'S wave, and my expectation of conical refraction which had not then been verified, Professor AIRY replied that he had long been aware of the existence of the conoidal cusps, which indeed it is surprising that FRESNEL did not perceive. Professor AIRY, however, had not perceived the existence of the circles of contact, nor had he drawn from either cusps or circles any theory of conical refraction.

The latter theory was deduced, by my general methods, from the hypothesis of transversal vibrations in a luminous ether, which hypothesis seems to have been first proposed by Dr. YOUNG, but to have been independently framed and far more perfectly developed by FRESNEL; and from FRESNEL'S other principle, of the existence of three rectangular axes of elasticity within a biaxal crystallised medium. The verification, therefore, of this theory of conical refraction, by the experiments of Professor LLOYD, must be considered as affording a new and important probability in favour of FRESNEL'S views: that is, a new encouragement to reason from those views, in combining and predicting appearances.

The length to which the present Supplement has already extended, obliges me to reserve, for a future communication, many other results deduced by my general methods from the principle of the characteristic function: and especially a general theory of the focal lengths and aberrations of optical instruments of revolution.

WILLIAM R. HAMILTON.

OBSERVATORY, *June 1833.*

CONTENTS OF THE THIRD SUPPLEMENT

10. General remarks and cautions, with respect to the foregoing deductions. Case of a single uniform medium. Connexions between the coefficients of the functions v, Ω, υ, for any single medium.

11. General formula for reflection or refraction, ordinary or extraordinary. Changes of V, W, T. The difference ΔV is $= 0$; $\Delta W = \Delta T =$ a homogeneous function of the first dimension of the differences $\Delta\sigma$, $\Delta\tau$, $\Delta\upsilon$, depending on the shape and position of the reflecting or refracting surface. Theorem of maxima and minima, for the elimination of the incident variables. Combinations of reflectors or refractors. Compound and component combinations.

12. Changes of the coefficients of the second order, of V, W, T, produced by reflection or refraction.

13. Changes produced by transformation of co-ordinates. Nearly all the foregoing results may be extended to oblique co-ordinates. The fundamental formula may be presented so as to extend even to polar or to any other marks of position, and new auxiliary functions may then be found, analogous to, and including, the functions W, T: together with new and general differential and integral equations for curved and polygon rays, ordinary and extraordinary.

14. General geometrical relations of infinitely near rays. Classification of twenty-four independent coefficients, which enter into the algebraical expressions of these general relations. Division of the general discussion into four principal problems.

15. Discussion of the four problems. Elements of arrangement of near luminous paths. Axis and constant of chromatic dispersion. Axis of curvature of ray. Guiding paraboloid, and constant of deviation. Guiding planes, and conjugate guiding axes.

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18. Construction of the new auxiliary paraboloid, (or of an osculating hyperboloid,) and of the new constant of deviation, for ray-lines from an oblique plane, by the former elements of arrangement.

19. Condition of intersection of two near final ray-lines. Conical locus of the near final points in a variable medium which satisfy this condition. Investigations of Malus. Illustration of the condition of intersection, by the theory of the auxiliary paraboloid, for ray-lines from an oblique plane.

20. Other geometrical illustrations of the condition of intersection, and of the elements of arrangement. Composition of partial deviations. Rotation round the axis of curvature of a final ray.

21. Relations between the elements of arrangement, depending only on the extreme

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22. Calculation of the elements of arrangement, for arbitrary axes of co-ordinates.

23. The general linear expressions for the arrangement of near rays, fail at a point of vergency. Determination of these points, and of their loci, the caustic surfaces, in a straight or curved system, by the methods of the present Supplement.

24. Connexion of the conditions of initial and final intersection of two near paths of light, polygon or curved, with the maxima or minima of the time or action-function $V + V_{\prime} = \sum \int v\, ds$. Separating planes, transition planes, and transition points, suggested by these maxima and minima. The separating planes divide the near points of less from those of greater action, and they contain the directions of osculation or intersection of the surfaces for which V and $V_{\prime}$ are constant; the transition planes touch the caustic pencils, and the transition points are on the caustic curves. Extreme osculating waves, or action surfaces. Law of osculation. Analogous theorems for sudden reflection or refraction.

25. Formulæ for the principal foci and principal rays of a straight or curved system, ordinary or extraordinary. General method of investigating the arrangement and aberrations of the rays, near a principal focus, or other point of vergency.

26. Combination of the foregoing view of optics with the undulatory theory of light. The quantities σ, τ, v, or

$$\frac{\delta V}{\delta x}, \frac{\delta V}{\delta y}, \frac{\delta V}{\delta z},$$

that is, the partial differential coefficients of the first order of the characteristic function V, taken with respect to the final co-ordinates, are, in the undulatory theory of light, the components of normal slowness of propagation of a wave. The fundamental formula (A) may easily be explained and proved by the principles of the same theory.

27. Theory of Fresnel. New formulae, founded on that theory, for the velocities and polarisations of a plane wave or wave-element. New method of deducing the equation of Fresnel's curved wave, propagated from a point in a uniform medium with three unequal elasticities. Lines of single ray-velocity, and of single normal-velocity, discovered by Fresnel.

28. New properties of Fresnel's wave. This wave has four conoidal cusps, at the ends of the lines of single ray-velocity; it has also four circles of contact, of which each is contained on a touching plane of single normal-velocity. The lines of single ray-velocity may therefore be called cusp-rays; and the lines of single normal-velocity may be called normals of circular contact.

29. New consequences of Fresnel's principles. It follows from those principles, that crystals of sufficient biaxal energy ought to exhibit two kinds of conical refraction, an external and an internal: a cusp-ray giving an external cone of rays, and a normal of circular contact being connected with an internal cone.

30. Theory of conical polarisation. Lines of vibration. These lines, on FRESNEL'S wave, are the intersections of two series of concentric and co-axal ellipsoids.

31. In any uniform medium, the curved wave propagated from a point is connected with a certain other surface, which may be called the surface of components, by relations discovered by M. CAUCHY, and by some new relations connected with a general theorem of reciprocity. This new theorem of reciprocity gives a new construction for the wave in any undulatory theory of light: and it connects the cusps and circles of contact on FRESNEL'S wave, with circles and cusps of the same kind upon the surface of components.

THIRD SUPPLEMENT

Fundamental Formula of Mathematical Optics. Design of the present Supplement.

1. When light is considered as propagated, according to that known general law which is called the law of least action, or of swiftest propagation, along any curved or polygon ray, ordinary or extraordinary, describing each element of that ray $ds = \sqrt{(dx^2 + dy^2 + dz^2)}$ with a molecular velocity or undulatory slowness v, which is supposed to depend, in the most general case, on the nature of the medium, the position and direction of the element, and the colour of the light, having only a finite number of values when these are given, and being therefore a function of the three rectangular co-ordinates, or marks of position, x, y, z, the three differential ratios or cosines of direction,

$$\alpha = \frac{dx}{ds}, \quad \beta = \frac{dy}{ds}, \quad \gamma = \frac{dx}{ds},$$

and a chromatic index or measure of colour, χ, the form of which function v depends on and characterises the medium; then if we denote as follows the variation of this function,

$$\delta v = \frac{\delta v}{\delta x}\delta x + \frac{\delta v}{\delta y}\delta y + \frac{\delta v}{\delta z}\delta z + \frac{\delta v}{\delta \alpha}\delta \alpha + \frac{\delta v}{\delta \beta}\delta \beta + \frac{\delta v}{\delta \gamma}\delta \gamma + \frac{\delta v}{\delta \chi}\delta \chi,$$

and if, by the help of the relation $\alpha^2 + \beta^2 + \gamma^2 = 1$, we determine

$$\frac{\delta v}{\delta \alpha}, \quad \frac{\delta v}{\delta \beta}, \quad \frac{\delta v}{\delta \gamma},$$

so as to satisfy the condition

$$\alpha \frac{\delta v}{\delta \alpha} + \beta \frac{\delta v}{\delta \beta} + \gamma \frac{\delta v}{\delta \gamma} = v,$$

namely, by making v homogeneous of the first dimension with respect to α, β, γ; it has been shown, in my First Supplement, that the variation of the definite integral $V = \int v\, ds$, considered as a function, which I have called the *Characteristic Function* of the final and initial co-ordinates, that is, the *variation of the action, or the time, expended by light of any one colour, in going from one variable point to another*, is

$$\delta V = (\delta \int v\, ds =)\ \frac{\delta v}{\delta \alpha}\delta x - \frac{\delta v'}{\delta \alpha'}\delta x' + \frac{\delta v}{\delta \beta}\delta y - \frac{\delta v'}{\delta \beta'}\delta y' + \frac{\delta v}{\delta \gamma}\delta z - \frac{\delta v'}{\delta \gamma'}\delta z' : \qquad \text{(A)}$$

the accented being the initial quantities. This general equation, (A), which I have called the *Equation of the Characteristic Function*, involves very various and extensive consequences, and appears to me to include the whole of mathematical optics. I propose, in the present Supplement, to offer some additional remarks and methods, connected with the characteristic function V, and the fundamental formula (A); and in particular to point out a new view of the auxiliary function W, introduced in my former memoirs, and a new auxiliary function T, which may be employed with advantage in many optical researches: I shall also give some other general transformations and applications of the fundamental formula, and shall speak of the connection of my view of optics with the undulatory theory of light.

Fundamental Problem of Mathematical Optics, and Solution by the Fundamental Formula. Partial Differential Equations, respecting the Characteristic Function V, and common to all optical combinations. Deduction of the Medium-Functions Ω, v, from this Characteristic Function V. Remarks on the new symbols σ, τ, υ.

2. It may be considered as a *fundamental problem* in Mathematical Optics, to which all others are reducible, *to determine, for any proposed combination of media, the law of dependence of the two extreme directions of a curved or polygon ray, ordinary or extraordinary, on the positions of the two extreme points which are visually connected by that ray, and on the colour of the light*: that is, in our present notation, to determine the law of dependence of the extreme *direction-cosines* $\alpha\ \beta\ \gamma\ \alpha'\ \beta'\ \gamma'$, on the extreme co-ordinates $x\ y\ z\ x'\ y'\ z'$, and on the chromatic index χ. This fundamental problem is resolved by our fundamental formula (A); or by the six following equations into which (A) resolves itself, and which express the law of dependence required:

$$\left.\begin{aligned} \frac{\delta V}{\delta x} &= \frac{\delta v}{\delta \alpha}; & \frac{\delta V}{\delta y} &= \frac{\delta v}{\delta \beta}; & \frac{\delta V}{\delta z} &= \frac{\delta v}{\delta \gamma}; \\ -\frac{\delta V}{\delta x'} &= \frac{\delta v'}{\delta \alpha'}; & -\frac{\delta V}{\delta y'} &= \frac{\delta v'}{\delta \beta'}; & -\frac{\delta V}{\delta z'} &= \frac{\delta v'}{\delta \gamma'}. \end{aligned}\right\} \tag{B}$$

These equations appear to require, for their application to any proposed combination, not only the knowledge of the form of the *Characteristic Function V*, that is, the law of dependence of the action or time on the extreme positions and on the colour, but also on the knowledge of the forms of the functions v, v', that is, the optical properties of the final and initial media; but these final and initial *medium-functions* v, v', may themselves be deduced from the one characteristic function V by reasonings of the following kind.

Whatever be the nature of the final medium, that is, whatever be the law of dependence of v on the position, direction, colour, we have supposed, in deducing the general formula (A), that the expression of this dependence has been so prepared as to make the medium-function v homogeneous of the first dimension relatively to the direction-cosines α, β, γ; the partial differential coefficients

$$\frac{\delta v}{\delta \alpha}, \quad \frac{\delta v}{\delta \beta}, \quad \frac{\delta v}{\delta \gamma},$$

of this homogeneous function, are themselves homogeneous, but of the dimension zero; that is, they are functions of the two ratios

$$\frac{\alpha}{\gamma}, \quad \frac{\beta}{\gamma},$$

involving also, in general, the co-ordinates $x\ y\ z$, and the chromatic index χ: if then we conceive the two ratios

$$\frac{\alpha}{\gamma}, \quad \frac{\beta}{\gamma},$$

to be eliminated between the three first of equations (B), and if, in like manner, we conceive

$$\frac{\alpha'}{\gamma'}, \quad \frac{\beta'}{\gamma'},$$

to be eliminated between the last three equations (B), we see that such eliminations would give two partial differential equations of the first order, between the characteristic function V and the co-ordinates and colour, of the form

$$\left.\begin{aligned} 0 &= \Omega\left(\frac{\delta V}{\delta x}, \frac{\delta V}{\delta y}, \frac{\delta V}{\delta z}, x, y, z, \chi\right), \\ 0 &= \Omega'\left(-\frac{\delta V}{\delta x'}, -\frac{\delta V}{\delta y'}, -\frac{\delta V}{\delta z'}, x', y', z', \chi\right), \end{aligned}\right\} \tag{C}$$

which both conduct to the following general equation, of the second order and third degree, common to all optical combinations,

$$\begin{aligned} &\frac{\delta^2 V}{\delta x\,\delta x'}\,\frac{\delta^2 V}{\delta y\,\delta y'}\,\frac{\delta^2 V}{\delta z\,\delta z'} + \frac{\delta^2 V}{\delta x\,\delta y'}\,\frac{\delta^2 V}{\delta y\,\delta z'}\,\frac{\delta^2 V}{\delta z\,\delta x'} + \frac{\delta^2 V}{\delta x\,\delta z'}\,\frac{\delta^2 V}{\delta y\,\delta x'}\,\frac{\delta^2 V}{\delta z\,\delta y'} \\ = &\frac{\delta^2 V}{\delta z\,\delta x'}\,\frac{\delta^2 V}{\delta y\,\delta y'}\,\frac{\delta^2 V}{\delta x\,\delta z'} + \frac{\delta^2 V}{\delta z\,\delta y'}\,\frac{\delta^2 V}{\delta y\,\delta z'}\,\frac{\delta^2 V}{\delta x\,\delta x'} + \frac{\delta^2 V}{\delta z\,\delta z'}\,\frac{\delta^2 V}{\delta y\,\delta x'}\,\frac{\delta^2 V}{\delta x\,\delta y'}. \end{aligned} \tag{D}$$

If now we put, for abridgment,

$$\left.\begin{aligned} \frac{\delta V}{\delta x} &= \sigma, & \frac{\delta V}{\delta y} &= \tau, & \frac{\delta V}{\delta z} &= \upsilon, \\ -\frac{\delta V}{\delta x'} &= \sigma', & -\frac{\delta V}{\delta y'} &= \tau', & -\frac{\delta V}{\delta z'} &= \upsilon', \end{aligned}\right\} \tag{E}$$

and if between the first three of these equations (E) we eliminate two of the three initial co-ordinates $x'\ y'\ z'$, it is easy to perceive, by (C) or (D), that in every optical combination the third co-ordinate will disappear; and similarly that between the three last equations (E) we can eliminate all the three final co-ordinates, by eliminating any two of them; and that these eliminations will conduct to the relations (C) under the form

$$\left.\begin{aligned} 0 &= \Omega(\sigma, \tau, \upsilon, x, y, z, \chi), \\ 0 &= \Omega'(\sigma', \tau', \upsilon', x', y', z', \chi), \end{aligned}\right\} \tag{F}$$

which can thus be obtained, by differentiation and elimination, from the characteristic function V alone: and which, as we are about to see, determine the forms of v, v', that is, the properties of the extreme media. Comparing the differentials of the relations (F), with the following, that is, with the conditions of homogeneity of v, v', prepared by the definitions (E) and by the relations (B),

$$\left.\begin{aligned} v &= \alpha\frac{\delta v}{\delta\alpha} + \beta\frac{\delta v}{\delta\beta} + \gamma\frac{\delta v}{\delta\gamma} = \alpha\sigma + \beta\tau + \gamma\upsilon, \\ v' &= \alpha'\frac{\delta v'}{\delta\alpha'} + \beta'\frac{\delta v'}{\delta\beta'} + \gamma'\frac{\delta v'}{\delta\gamma'} = \alpha'\sigma' + \beta'\tau' + \gamma'\upsilon', \end{aligned}\right\} \tag{G}$$

and with their differentials, that is with

$$\left.\begin{aligned} \alpha\,\delta\sigma + \beta\,\delta\tau + \gamma\,\delta\upsilon &= \frac{\delta v}{\delta x}\delta x + \frac{\delta v}{\delta y}\delta y + \frac{\delta v}{\delta z}\delta z + \frac{\delta v}{\delta\chi}\delta\chi, \\ \alpha'\,\delta\sigma' + \beta'\,\delta\tau' + \gamma'\,\delta\upsilon' &= \frac{\delta v'}{\delta x'}\delta x' + \frac{\delta v'}{\delta y'}\delta y' + \frac{\delta v'}{\delta z'}\delta z' + \frac{\delta v'}{\delta\chi}\delta\chi, \end{aligned}\right\} \tag{H}$$

we find

$$\left.\begin{aligned} &\frac{\alpha}{v} = \frac{\delta\Omega}{\delta\sigma}, \qquad \frac{\beta}{v} = \frac{\delta\Omega}{\delta\tau}, \qquad \frac{\gamma}{v} = \frac{\delta\Omega}{\delta\upsilon}, \\ &\frac{\alpha'}{v'} = \frac{\delta\Omega'}{\delta\sigma'}, \qquad \frac{\beta'}{v'} = \frac{\delta\Omega'}{\delta\tau'}, \qquad \frac{\gamma'}{v'} = \frac{\delta\Omega'}{\delta\upsilon'}, \end{aligned}\right\} \tag{I}$$

and also

$$\left.\begin{aligned} &-\frac{1}{v}\frac{\delta v}{\delta x} = \frac{\delta\Omega}{\delta x}, \qquad -\frac{1}{v}\frac{\delta v}{\delta y} = \frac{\delta\Omega}{\delta y}, \qquad -\frac{1}{v}\frac{\delta v}{\delta z} = \frac{\delta\Omega}{\delta z}, \qquad -\frac{1}{v}\frac{\delta v}{\delta\chi} = \frac{\delta\Omega}{\delta\chi}, \\ &-\frac{1}{v'}\frac{\delta v'}{\delta x'} = \frac{\delta\Omega'}{\delta x'}, \qquad -\frac{1}{v'}\frac{\delta v'}{\delta y'} = \frac{\delta\Omega'}{\delta y'}, \qquad -\frac{1}{v'}\frac{\delta v'}{\delta z'} = \frac{\delta\Omega'}{\delta z'}, \qquad -\frac{1}{v'}\frac{\delta v'}{\delta\chi} = \frac{\delta\Omega'}{\delta\chi}, \end{aligned}\right\} \tag{K}$$

if we so prepare the expressions of the relations (F) as to have

$$\left.\begin{aligned} \sigma\frac{\delta\Omega}{\delta\sigma} + \tau\frac{\delta\Omega}{\delta\tau} + \upsilon\frac{\delta\Omega}{\delta\upsilon} &= 1, \\ \sigma'\frac{\delta\Omega'}{\delta\sigma'} + \tau'\frac{\delta\Omega'}{\delta\tau'} + \upsilon'\frac{\delta\Omega'}{\delta\upsilon'} &= 1; \end{aligned}\right\} \tag{L}$$

which can be done by putting those relations under the form

$$\left.\begin{aligned} 0 &= (\sigma^2 + \tau^2 + \upsilon^2)^{\frac{1}{2}}\omega - 1 = \Omega, \\ 0 &= (\sigma'^2 + \tau'^2 + \upsilon'^2)^{\frac{1}{2}}\omega' - 1 = \Omega'; \end{aligned}\right\} \tag{M}$$

in which ω, ω', that is $(\sigma^2 + \tau^2 + \upsilon^2)^{-\frac{1}{2}}$, and $(\sigma'^2 + \tau'^2 + \upsilon'^2)^{-\frac{1}{2}}$, are to be expressed as functions respectively of $\sigma(\sigma^2 + \tau^2 + \upsilon^2)^{-\frac{1}{2}}$, $\tau(\sigma^2 + \tau^2 + \upsilon^2)^{-\frac{1}{2}}$, $\upsilon(\sigma^2 + \tau^2 + \upsilon^2)^{-\frac{1}{2}}$, x, y,

z, χ, and of $\sigma'(\sigma'^2 + \tau'^2 + \upsilon'^2)^{-\frac{1}{2}}$, $\tau'(\sigma'^2 + \tau'^2 + \upsilon'^2)^{-\frac{1}{2}}$, $\upsilon'(\sigma'^2 + \tau'^2 + \upsilon'^2)^{-\frac{1}{2}}$, x', y', z', χ. After this preparation the partial differential coefficients

$$\frac{\delta\Omega}{\delta\sigma}, \quad \frac{\delta\Omega}{\delta\tau}, \quad \frac{\delta\Omega}{\delta\upsilon},$$

are homogeneous of dimension zero relatively to σ, τ, υ; and in like manner

$$\frac{\delta\Omega'}{\delta\sigma'}, \quad \frac{\delta\Omega'}{\delta\tau'}, \quad \frac{\delta\Omega'}{\delta\upsilon'},$$

are homogeneous of dimension zero relatively to σ', τ', υ'; if, therefore, between the three first equations (I), we eliminate any two of the three final quantities σ, τ, υ, the third will disappear; and similarly all the three initial quantities σ', τ', υ', can be eliminated together, between the last three of the equations (I): and by these eliminations we shall be conducted to two relations of the form

$$\left.\begin{aligned} 0 &= \Psi(\frac{\alpha}{v}, \frac{\beta}{v}, \frac{\gamma}{v}, x, y, z, \chi), \\ 0 &= \Psi'(\frac{\alpha'}{v'}, \frac{\beta'}{v'}, \frac{\gamma'}{v'}, x', y', z', \chi), \end{aligned}\right\} \tag{N}$$

which determine the forms of the final and initial medium-functions v, v'; so that these forms can be deduced from the form of the characteristic function V. We can therefore reduce to the study of this one function V, that general problem of mathematical optics which has been already mentioned.

The partial differential coefficients of the characteristic function V, taken with respect to the co-ordinates x, y, z, are of continual occurrence in the optical methods of my present and former memoirs; I have therefore thought it useful to denote them in this Supplement by separate symbols σ, τ, υ, and I shall show in a future number their meanings in the undulatory theory: namely, that they denote, in it, the components of normal slowness of propagation of a wave.

Connection of the Characteristic Function V, with the Formation and Integration of the General Equations of a Curved Ray, Ordinary or Extraordinary.

3. It may be considered as a particular case of the foregoing general problem, to determine general forms for the differential equations of a curved ray, ordinary or extraordinary; that is, to connect the general changes of direction with those of position, in the passage of light through a variable medium. The following forms,

$$d\frac{\delta v}{\delta\alpha} = \frac{\delta v}{\delta x}ds, \quad d\frac{\delta v}{\delta\beta} = \frac{\delta v}{\delta y}ds, \quad d\frac{\delta v}{\delta\gamma} = \frac{\delta v}{\delta z}ds, \tag{O}$$

(which are of the second order, because α, β, γ, α', β', γ', are defined by the equations

$$\left.\begin{aligned} \alpha &= \frac{dx}{ds}, & \beta &= \frac{dy}{ds}, & \gamma &= \frac{dz}{ds}, \\ \alpha' &= \frac{dx'}{ds'}, & \beta' &= \frac{dy'}{ds'}, & \gamma' &= \frac{dz'}{ds'}, \end{aligned}\right\} \tag{P}$$

the symbol d referring, throughout the present Supplement, to motion along a ray, while δ *refers to arbitrary infinitesimal changes of position, direction, and colour*, and ds' being the initial element of the ray,) were deduced in the First Supplement, by the Calculus of Variations, from the law of least action. The same forms (O), which are equivalent to but two distinct equations, may be deduced from the fundamental formula (A), by the properties of the characteristic function V. For if we differentiate the first equation (C), (which involves the coefficients of this function V, and was deduced from the formula (A),) with reference to each of the three co-ordinates x, y, z, considered as three independent variables, and with reference to the index of colour χ, we find, by the foregoing number,

$$\left.\begin{aligned}
\alpha\frac{\delta^2 V}{\delta x^2} + \beta\frac{\delta^2 V}{\delta x\,\delta y} + \gamma\frac{\delta^2 V}{\delta x\,\delta z} &= \frac{\delta v}{\delta x}, \\
\alpha\frac{\delta^2 V}{\delta x\,\delta y} + \beta\frac{\delta^2 V}{\delta y^2} + \gamma\frac{\delta^2 V}{\delta y\,\delta z} &= \frac{\delta v}{\delta y}, \\
\alpha\frac{\delta^2 V}{\delta x\,\delta z} + \beta\frac{\delta^2 V}{\delta y\,\delta z} + \gamma\frac{\delta^2 V}{\delta z^2} &= \frac{\delta v}{\delta z}, \\
\alpha\frac{\delta^2 V}{\delta x\,\delta \chi} + \beta\frac{\delta^2 V}{\delta y\,\delta \chi} + \gamma\frac{\delta^2 V}{\delta z\,\delta \chi} &= \frac{\delta v}{\delta \chi},
\end{aligned}\right\} \tag{Q}$$

and the three first of the equations (Q), by the general relations (B), which were themselves deduced from (A), and by the meanings (P) of α, β, γ, may easily be transformed to (O). The differential equations (O) may also be regarded as the limits of the following,

$$\sigma - \sigma' = \left(\frac{\delta V}{\delta x}\right), \quad \tau - \tau' = \left(\frac{\delta V}{\delta y}\right), \quad \upsilon - \upsilon' = \left(\frac{\delta V}{\delta z}\right), \tag{R}$$

in which

$$\left(\frac{\delta V}{\delta x}\right), \quad \left(\frac{\delta V}{\delta y}\right), \quad \left(\frac{\delta V}{\delta z}\right),$$

are obtained by differentiating V considered as a function of the seven variables x, y, z, Δx, Δy, Δz, χ, if $\Delta x = x - x'$, $\Delta y = y - y'$, $\Delta z = z - z'$; the variation of V, when so considered, being by (A), and by the definitions (E),

$$\begin{aligned}
\delta V = {} & (\sigma - \sigma')\delta x + (\tau - \tau')\delta y + (\upsilon - \upsilon')\delta z \\
& + \left(\frac{\delta V}{\delta \Delta x}\right)\delta\Delta x + \left(\frac{\delta V}{\delta \Delta y}\right)\delta\Delta y + \left(\frac{\delta V}{\delta \Delta z}\right)\delta\Delta z + \frac{\delta V}{\delta \chi}\delta\chi,
\end{aligned} \tag{S}$$

in which

$$\left(\frac{\delta V}{\delta \Delta x}\right) = \sigma', \quad \left(\frac{\delta V}{\delta \Delta y}\right) = \tau', \quad \left(\frac{\delta V}{\delta \Delta z}\right) = \upsilon'. \tag{T}$$

If we differentiate the first equation (C) relatively to x', y', z', we find, by the foregoing number,

$$\left.\begin{aligned}
0 &= \alpha\frac{\delta^2 V}{\delta x\,\delta x'} + \beta\frac{\delta^2 V}{\delta y\,\delta x'} + \gamma\frac{\delta^2 V}{\delta z\,\delta x'},\\
0 &= \alpha\frac{\delta^2 V}{\delta x\,\delta y'} + \beta\frac{\delta^2 V}{\delta y\,\delta y'} + \gamma\frac{\delta^2 V}{\delta z\,\delta y'},\\
0 &= \alpha\frac{\delta^2 V}{\delta x\,\delta z'} + \beta\frac{\delta^2 V}{\delta y\,\delta z'} + \gamma\frac{\delta^2 V}{\delta z\,\delta z'},
\end{aligned}\right\} \tag{U}$$

of which, in virtue of (D), any two include the third, and which may be put by (P) under the form

$$0 = d\frac{\delta V}{\delta x'};\quad 0 = d\frac{\delta V}{\delta y'};\quad 0 = d\frac{\delta V}{\delta z'}; \tag{V}$$

and these differential equations (V) of the first order, in which the initial co-ordinates and the colour are constant, belong to the ray, and may be regarded as integrals of (O). They have, themselves, for integrals

$$\frac{\delta V}{\delta x'} = \text{const.},\quad \frac{\delta V}{\delta y'} = \text{const.},\quad \frac{\delta V}{\delta z'} = \text{const.}, \tag{W}$$

the constants being, by (B), the values of the initial quantities

$$-\frac{\delta v'}{\delta\alpha'},\quad -\frac{\delta v'}{\delta\beta'},\quad -\frac{\delta v'}{\delta\gamma'}.$$

In like manner, by differentiating the last equation (C), we find the following equations, which are analogous to (Q) and (U),

$$\left.\begin{aligned}
\alpha'\frac{\delta^2 V}{\delta x'^2} + \beta'\frac{\delta^2 V}{\delta x'\,\delta y'} + \gamma'\frac{\delta^2 V}{\delta x'\,\delta z'} &= -\frac{\delta v'}{\delta x'},\\
\alpha'\frac{\delta^2 V}{\delta x'\,\delta y'} + \beta'\frac{\delta^2 V}{\delta y'^2} + \gamma'\frac{\delta^2 V}{\delta y'\,\delta z'} &= -\frac{\delta v'}{\delta y'},\\
\alpha'\frac{\delta^2 V}{\delta x'\,\delta z'} + \beta'\frac{\delta^2 V}{\delta y'\,\delta z'} + \gamma'\frac{\delta^2 V}{\delta z'^2} &= -\frac{\delta v'}{\delta z'},\\
\alpha'\frac{\delta^2 V}{\delta x'\,\delta \chi} + \beta'\frac{\delta^2 V}{\delta y'\,\delta \chi} + \gamma'\frac{\delta^2 V}{\delta z'\,\delta \chi} &= -\frac{\delta v'}{\delta \chi},
\end{aligned}\right\} \tag{X}$$

and

$$\left.\begin{aligned}
0 &= \alpha'\frac{\delta^2 V}{\delta x\,\delta x'} + \beta'\frac{\delta^2 V}{\delta x\,\delta y'} + \gamma'\frac{\delta^2 V}{\delta x\,\delta z'},\\
0 &= \alpha'\frac{\delta^2 V}{\delta y\,\delta x'} + \beta'\frac{\delta^2 V}{\delta y\,\delta y'} + \gamma'\frac{\delta^2 V}{\delta y\,\delta z'},\\
0 &= \alpha'\frac{\delta^2 V}{\delta z\,\delta x'} + \beta'\frac{\delta^2 V}{\delta z\,\delta y'} + \gamma'\frac{\delta^2 V}{\delta z\,\delta z'},
\end{aligned}\right\} \tag{Y}$$

The second members of the three first equations (X) vanish when the initial medium is uniform, and those of the three first equations (Q) when the final medium is so; and in this latter case, of a final uniform medium, the final portion of the ray is straight, and in its whole extent we have not only the equations (W) but also the following,

$$\frac{\delta V}{\delta x} = \text{const.}, \quad \frac{\delta V}{\delta y} = \text{const.}, \quad \frac{\delta V}{\delta z} = \text{const.}, \tag{Z}$$

the constants being by (B) those functions of the final direction-cosines and of the colour which we have denoted by

$$\frac{\delta v}{\delta \alpha}, \quad \frac{\delta v}{\delta \beta}, \quad \frac{\delta v}{\delta \gamma},$$

and which are here independent of the co-ordinates. In general, if we consider the final co-ordinates and the colour as constant, the relations (Z) between the initial co-ordinates are forms for the equations of a ray. And though we have hitherto considered rectangular co-ordinates only, yet we shall show in a future number that there are analogous results of oblique and even for polar co-ordinates.

Combination of the foregoing View of Optics with the Undulatory Theory of Light. The quantities σ, τ, υ, or $\dfrac{\delta V}{\delta x}$, $\dfrac{\delta V}{\delta y}$, $\dfrac{\delta V}{\delta z}$, that is, the Partial Differential Coefficients of the First Order of the Characteristic Function V, taken with respect to the Final Coordinates, are, in the Undulatory Theory of Light, the Components of Normal Slowness of Propagation of a Wave. The Fundamental Formula (A) *may easily be explained and proved by the principles of the same theory.*

26. It remains, for the execution of the design announced at the beginning of this Supplement, to illustrate the mathematical view of optics proposed in this and in former memoirs, by connecting it more closely with the undulatory theory of light.

For this purpose we shall begin by examining the undulatory meanings of the symbols σ, τ, υ, of which, in the present Supplement, we have made so frequent a use, and which we have defined by the equations (E),

$$\sigma = \frac{\delta V}{\delta x}, \quad \tau = \frac{\delta V}{\delta y}, \quad \upsilon = \frac{\delta V}{\delta z},$$

V being the undulatory time of propagation of light of some given colour, from some origin x', y', z', to a point x, y, z through any combination of media. It is evident that these quantities σ, τ, υ are proportional to the direction-cosines of the normal to the wave for which the time V is constant, and which has for its differential equation

$$0 = \delta V = \sigma\,\delta x + \tau\,\delta y + \upsilon\,\delta z; \tag{A^{18}}$$

and if, as in the second number, we denote $(\sigma^2 + \tau^2 + \upsilon^2)^{-\frac{1}{2}}$ by ω, these direction-cosines themselves will be $\sigma\omega$, $\tau\omega$, $\upsilon\omega$; and ω will be the *normal velocity*, because the infinitesimal time δV, during which the wave propagates itself in the direction of its own normal through the infinitesimal line δl, from the point x, y, z, to the point $x + \sigma\omega.\delta l$, $y + \tau\omega.\delta l$, $z + \upsilon\omega.\delta l$, is

$$\delta V = \sigma.\sigma\omega.\delta l + \tau.\tau\omega.\delta l, + \upsilon.\upsilon\omega.\delta l = \frac{1}{\omega}\,\delta l: \tag{B^{18}}$$

we may therefore call the quantities σ, τ, υ, *the components of normal slowness*, because they are equal to the reciprocal of the normal velocity, that is, to *the normal slowness, multiplied respectively by the direction-cosines of the normal*, that is, by the cosines of the angles which it makes with the rectangular axes of co-ordinates.

Such them may be said to be the optical meaning of our quantities σ, τ, υ, in the theory of the propagation of light by waves. And we might easily deduce from this meaning, and from the first principles of the undulatory theory, the general expression (A) for the variation of the characteristic function V, which has been proposed in the present and former memoirs, as fundamental in mathematical optics. For it is an immediate consequence of the dynamical ideas of the undulatory theory of light, that for a plane wave of a given direction and colour, in a given uniform medium, the normal velocity of propagation is determined, or at least restricted to a finite variety of values; so that this normal velocity may be considered as a function of its cosines of direction, involving also the colour, and depending for its form on the nature of the uniform medium, and on the positions of the axes of co-ordinates, to which the angles of direction are referred: and if the medium be variable instead of uniform, and the wave curved instead of plane, we must suppose that the normal velocity ω is still a function of its direction-cosines $\sigma(\sigma^2+\tau^2+\upsilon^2)^{-\frac{1}{2}}$, $\tau(\sigma^2+\tau^2+\upsilon^2)^{-\frac{1}{2}}$, $\upsilon(\sigma^2+\tau^2+\upsilon^2)^{-\frac{1}{2}}$, and of the colour χ, involving also, in this more general case, the co-ordinates x, y, z. In this manner we are conducted, by the principles of the undulatory theory, to a relation between σ, τ, υ, x, y, z, χ, of the kind already often employed in the present Supplement, namely,

$$0 = \Omega = (\sigma^2+\tau^2+\upsilon^2)^{\frac{1}{2}}\omega - 1, \tag{M}$$

$\Omega + 1$ being a homogeneous function of σ, τ, υ, of the first dimension, which satisfies therefore the condition

$$\sigma\frac{\delta\Omega}{\delta\sigma} + \tau\frac{\delta\Omega}{\delta\tau} + \upsilon\frac{\delta\Omega}{\delta\upsilon} = \Omega + 1,$$

and which involves also in general the co-ordinates x, y, z, and the colour χ, and depends for its form on the optical properties of the medium in which the point $x\ y\ z$ is placed. *To connect now*, for any given point and colour, the *velocity and direction of the ray with the direction of the normal of the wave*, we may suppose, at first, that the medium is uniform, and that the wave is plane. The two positions of this plane wave, at the time V, and at the time $V+\Delta V$, may be denoted by the equations

$$\left.\begin{array}{ll} \text{First} & \sigma x + \tau y + \upsilon z = V + W, \\ \text{Second} & \sigma\Delta x + \tau\Delta y + \upsilon\Delta z = \Delta V, \end{array}\right\} \tag{C^{18}}$$

in which σ, τ, υ, W, are constants; and by the principles of the same undulatory theory, if the point $x+\Delta x$, $y+\Delta y$, $z+\Delta z$, on the second plane wave, corresponding to the time $V+\Delta V$, be upon the ray that passes through the point $x\ y\ z$ of the first plane wave, it will be also on all the other infinitely near plane waves which correspond to the same time $V+\Delta V$, these other waves having passed through the point $x\ y\ z$ at the time V, and having made infinitely small angles with the first plane wave; we are therefore to find the co-ordinates $x+\Delta x$, $y+\Delta y$, $z+\Delta z$, of the second point upon the ray, by seeking the intersection of the second wave (C^{18}) with all those other waves which are obtained from it by assigning to σ, τ, υ, any infinitely small variations consistent with the relation

$$0 = \delta\Omega = \frac{\delta\Omega}{\delta\sigma}\delta\sigma + \frac{\delta\Omega}{\delta\tau}\delta\tau + \frac{\delta\Omega}{\delta\upsilon}\delta\upsilon;$$

and thus we find

$$\frac{\alpha}{v} = \frac{\Delta x}{\Delta V} = \frac{\delta\Omega}{\delta\sigma}, \quad \frac{\beta}{v} = \frac{\Delta y}{\Delta V} = \frac{\delta\Omega}{\delta\tau}, \quad \frac{\gamma}{v} = \frac{\Delta z}{\Delta V} = \frac{\delta\Omega}{\delta\upsilon}, \tag{D^{18}}$$

as in the second number of this Supplement, and therefore

$$\begin{aligned} v &= \alpha\sigma + \beta\tau + \gamma\upsilon, \\ 0 &= \alpha\,\delta\sigma + \beta\,\delta\tau + \gamma\,\delta\upsilon, \\ \delta v &= \sigma\,\delta\alpha + \tau\,\delta\beta + \upsilon\,\delta\gamma, \end{aligned}$$

and finally

$$\frac{\delta v}{\delta\alpha} = \sigma, \quad \frac{\delta v}{\delta\beta} = \tau, \quad \frac{\delta v}{\delta\gamma} = \upsilon, \tag{E^{18}}$$

if we denote by v the reciprocal of the undulatory velocity with which the light is propagated along the ray, and by α, β, γ, the cosines of the angles which the ray makes with the axes of co-ordinates. We see, therefore, by the foregoing reasoning, which it is easy to extend to the case of curved waves and of variable media, that *the components σ, τ, υ, of normal slowness of a wave, or the partial differential coefficients of the first order of the time-function V, are equal to the partial differential coefficients of the first order, $\frac{\delta v}{\delta\alpha}$, $\frac{\delta v}{\delta\beta}$, $\frac{\delta v}{\delta\gamma}$, of the undulatory slowness V of propagation along the ray, when this latter slowness is expressed as a homogeneous function of the first dimension of the direction-cosines α β γ of the ray*: which is the general theorem of mathematical optics, expressed by our fundamental formula (A).

That general theorem does not appear to have been perceived by other writers; nor do they seem to have distinctly thought of the components of normal slowness, nor of the function of which those components are partial differential coefficients, that is, the time V of propagation of light from one variable point to another, through any combination of uniform or variable media, considered as depending on the final and initial co-ordinates and on the colour: much less do those who have hitherto written upon light, appear to have thought of *the time-function V as a* CHARACTERISTIC FUNCTION, *to the study of which may be reduced all the problems of mathematical optics.* But the problem of connecting by general equations the direction and velocity of a ray with the direction and with the law of normal velocity of a wave, has been elegantly resolved by M. CAUCHY, in the 50th *Livraison* of the *Exercices de Mathématiques*: and the formulæ which have been there deduced by considering the normal velocity as a homogeneous function of the first dimension of its three cosines of direction, may easily be shown to agree with the equations (D^{18}).

Theory of FRESNEL. *New Formulæ, founded on that theory, for the Velocities and Polarisations of a Plane Wave or Wave-Element. New method of deducing the Equation of* FRESNEL'S *Curved Wave, propagated from a Point in a Uniform Medium with Three Unequal Elasticities. Lines of Single Ray-Velocity, and of Single Normal-Velocity, discovered by* FRESNEL.

27. Let us now consider more particularly the undulatory theory of FRESNEL.

In that theory, the small displacements of the vibrating etherial points are confined to the surface of the wave, the ether being supposed to be sensibly incompressible, and so to resist and prevent any sensible normal vibration: and the tangential forces, which regulate the tangential or transversal vibrations, result in general from the elasticity of the ether, combined with this normal resistance. It is also supposed that the etherial medium has in general three principal unequal elasticities, corresponding to displacements in the directions of three rectangular *axes of elasticity*; in such a manner that if we take these for the axes of co-ordinates, any small component displacements δx, δy, δz parallel to these three axes will produce elastic forces $-a^2\,\delta x$, $-b^2\delta y$, $-c^2\,\delta z$ parallel to the same axes, and equal to the displacements taken with contrary signs and multiplied by certain constant positive factors a^2, b^2, c^2; and any small resultant displacement, δl, in any other direction, having δx, δy, δz for its components or projections, will produce a corresponding elastic force $-E\,\delta l$, of which the components are $-a^2\,\delta x$, $-b^2\,\delta y$, $-c^2\,\delta z$, and which has not in general the same direction as the displacement δl, nor a direction exactly opposite to that. Light, polarised in any plane, P, is suppose to correspond to vibrations perpendicular to that plane, and propagated without change of direction; and in order that a vibration should thus preserve its direction unchanged, while the plane wave or wave-element to which it belongs is propagated through the uniform medium with a normal velocity ω, it is necessary and sufficient that the elastic force $-E\,\delta l$, when combined with a normal resistance arising from the incompressibility of the ether, should produce a tangential force $-\omega^2\,\delta l$, in the direction opposite to the displacement δl, and equal to this displacement taken with a contrary sign, and multiplied by the square of the normal velocity of propagation, so that its components are $-\omega^2\,\delta x$, $-\omega^2\,\delta y$, $-\omega^2\,\delta z$: that is, we must have the equations

$$\frac{1}{\sigma}(\omega^2 - a^2)\delta x = \frac{1}{\tau}(\omega^2 - b^2)\delta y = \frac{1}{\upsilon}(\omega^2 - c^2)\delta z, \tag{F^{18}}$$

in which σ, τ, υ, are, as before, the components of normal slowness, so that the equation of the wave-element containing the transversal vibration is

$$\sigma\,\delta x + \tau\,\delta y + \upsilon\,\delta z = 0. \tag{A^{18}}$$

These equations (A^{18}) (F^{18}) *suffice in general to determine, on* FRESNEL'S *principles, the velocities of propagation and the planes of polarisation for any given wave-element in any known crystallised medium.*

Thus, eliminating the components of displacement δx, δy, δz, between the equations (A^{18}) (F^{18}), we find the following *law of the normal velocity* ω, considered as depending on the normal direction, that is, on the ratios of σ, τ, υ,

$$\frac{\sigma^2}{\omega^2 - a^2} + \frac{\tau^2}{\omega^2 - b^2} + \frac{\upsilon^2}{\omega^2 - c^2} = 0. \tag{G^{18}}$$

To deduce hence the *direction and velocity of a ray, for any given normal direction and normal velocity*, compatible with the foregoing law, that is, for any given values of the components of normal slowness σ, τ, υ, compatible with the relation (G^{18}), we are to make, by (M),

$$\omega^2 = \frac{(\Omega + 1)^2}{\sigma^2 + \tau^2 + \upsilon^2}, \tag{H^{18}}$$

and we then find, by (I), or by (D^{18}), the following expressions for the components of the velocity of the ray,

$$\left.\begin{aligned}
\frac{\alpha}{v} &= \frac{\delta\Omega}{\delta\sigma} = \frac{\sigma\omega^2}{\Omega + 1}\,\frac{\lambda^2 - a^2}{\omega^2 - a^2},\\
\frac{\beta}{v} &= \frac{\delta\Omega}{\delta\tau} = \frac{\tau\omega^2}{\Omega + 1}\,\frac{\lambda^2 - b^2}{\omega^2 - b^2},\\
\frac{\gamma}{v} &= \frac{\delta\Omega}{\delta\upsilon} = \frac{\upsilon\omega^2}{\Omega + 1}\,\frac{\lambda^2 - c^2}{\omega^2 - c^2},
\end{aligned}\right\} \tag{I^{18}}$$

if we put for abridgment

$$\lambda^2 = \frac{\left(\frac{a^2\sigma}{\omega^2 - a^2}\right)^2 + \left(\frac{b^2\tau}{\omega^2 - b^2}\right)^2 + \left(\frac{c^2\upsilon}{\omega^2 - c^2}\right)^2}{\left(\frac{a\sigma}{\omega^2 - a^2}\right)^2 + \left(\frac{b\tau}{\omega^2 - b^2}\right)^2 + \left(\frac{c\upsilon}{\omega^2 - c^2}\right)^2}. \tag{K^{18}}$$

And to deduce the *law of the velocity* $\frac{1}{v}$ *of the ray, considered as depending on its own direction*, that is, on the cosines $\alpha\ \beta\ \gamma$ of its inclinations to the semiaxes $a\ b\ c$ of elasticity, we are to eliminate (according to the general method of the second number) the ratios of $\sigma\ \tau\ \upsilon$ between the three expressions (I^{18}), and so deduce the relations between the three components of velocity $\frac{\alpha}{v}, \frac{\beta}{v}, \frac{\gamma}{v}$; now the equations (I^{18}) given evidently, by (K^{18}),

$$\frac{a^2\alpha^2}{\lambda^2 - a^2} + \frac{b^2\beta^2}{\lambda^2 - b^2} + \frac{c^2\gamma^2}{\lambda^2 - c^2} = 0; \tag{L^{18}}$$

they give also, when we attend to (G^{18}),

$$\left(\frac{\alpha}{v}\right)^2 + \left(\frac{\beta}{v}\right)^2 + \left(\frac{\gamma}{v}\right)^2 = \lambda^2 : \tag{M^{18}}$$

λ therefore is the velocity of the ray, or the radius vector of the curved *unit-wave*, propagated in all directions from the origin of co-ordinates during the unit of time; and the *equation of the wave* in rectangular co-ordinates $x\ y\ z$, parallel to the axes of elasticity, is

$$\frac{a^2x^2}{x^2 + y^2 + z^2 - a^2} + \frac{b^2y^2}{x^2 + y^2 + z^2 - b^2} + \frac{b^2z^2}{x^2 + y^2 + z^2 - c^2} = 0, \tag{N^{18}}$$

or, when freed from fractions

$$(x^2 + y^2 + z^2)(a^2x^2 + b^2y^2 + c^2z^2) + a^2b^2c^2 = a^2(b^2 + c^2)x^2 + b^2(c^2 + a^2)y^2 + c^2(a^2 + b^2)z^2. \tag{O^{18}}$$

This method of determining the equation of Fresnel's *Wave*, will perhaps be thought simpler than that which was employed by the illustrious discover, and than others which have since been proposed.

Reciprocally to determine by our general methods the normal direction and velocity, or the components of normal slowness σ, τ, υ, for any proposed direction and velocity of a ray compatible with this form of the wave, that is, for any values of $\alpha\ \beta\ \gamma\ \lambda$ compatible with the relation (L^{18}), we are to substitute for the ray-velocity λ in that relation its value (M^{18}), and we find, by (E^{18}),

$$\left.\begin{aligned} \sigma &= \frac{\delta v}{\delta\alpha} = \frac{\alpha}{v}.\frac{1 - a^2\nu^2}{\lambda^2 - a^2}, \\ \tau &= \frac{\delta v}{\delta\beta} = \frac{\beta}{v}.\frac{1 - b^2\nu^2}{\lambda^2 - b^2}, \\ \upsilon &= \frac{\delta v}{\delta\gamma} = \frac{\gamma}{v}.\frac{1 - c^2\nu^2}{\lambda^2 - c^2}, \end{aligned}\right\} \tag{P^{18}}$$

if we put for abridgment

$$\nu^2 = \frac{\left(\frac{\alpha}{\lambda^2 - a^2}\right)^2 + \left(\frac{\beta}{\lambda^2 - b^2}\right)^2 + \left(\frac{\gamma}{\lambda^2 - c^2}\right)^2}{\left(\frac{a\alpha}{\lambda^2 - a^2}\right)^2 + \left(\frac{b\beta}{\lambda^2 - b^2}\right)^2 + \left(\frac{c\gamma}{\lambda^2 - c^2}\right)^2}. \qquad (\mathrm{Q}^{18})$$

It is easy to see that the value of ν thus determined is the normal slowness, or reciprocal of ω, because the expressions (P^{18}) give, by (L^{18}),

$$\sigma^2 + \tau^2 + \upsilon^2 = \nu^2; \qquad (\mathrm{R}^{18})$$

and since the same expressions given also evidently, by (Q^{18}),

$$\frac{\sigma^2}{1 - a^2\nu^2} + \frac{\tau^2}{1 - b^2\nu^2} + \frac{\upsilon^2}{1 - c^2\nu^2} = 0, \qquad (\mathrm{S}^{18})$$

we easily deduce the law (G^{18}) of dependence of the normal velocity on the normal direction, from the form of FRESNEL'S wave, as we had deduced the latter from the former.

The equations (L^{18}) (M^{18}) which gave us the equation of the wave in rectangular co-ordinates, give also the following polar equation for the reciprocal of its radius-vector, that is, for the slowness v of the ray,

$$\begin{aligned} 0 = v^4 &- v^2\{\alpha^2(b^{-2} + c^{-2}) + \beta^2(c^{-2} + a^{-2}) + \gamma^2(a^{-2} + b^{-2})\} \\ &+ (\alpha^2 + \beta^2 + \gamma^2)(\alpha^2 b^{-2} c^{-2} + \beta^2 c^{-2} a^{-2} + \gamma^2 a^{-2} b^{-2}), \end{aligned} \qquad (\mathrm{T}^{18})$$

and therefore the following double expression for the square of this slowness

$$\begin{aligned} v^2 = &\tfrac{1}{2}(c^{-2} + a^{-2})(\alpha^2 + \beta^2 + \gamma^2) \\ &+ \tfrac{1}{2}(c^{-2} - a^{-2})\{A'A'' \pm \sqrt{\alpha^2 + \beta^2 + \gamma^2 - A'^2}\sqrt{\alpha^2 + \beta^2 + \gamma^2 - A''^2}\}, \end{aligned} \qquad (\mathrm{U}^{18})$$

if we put for abridgment

$$\left.\begin{aligned} A' &= \alpha\sqrt{\frac{b^{-2} - a^{-2}}{c^{-2} - a^{-2}}} + \gamma\sqrt{\frac{c^{-2} - b^{-2}}{c^{-2} - a^{-2}}}, \\ A'' &= \alpha\sqrt{\frac{b^{-2} - a^{-2}}{c^{-2} - a^{-2}}} - \gamma\sqrt{\frac{c^{-2} - b^{-2}}{c^{-2} - a^{-2}}} : \end{aligned}\right\} \qquad (\mathrm{V}^{18})$$

supposing therefore $a^2 > b^2 > c^2$, the *polar equation of the wave* may be put under the form

$$\rho^{-2} = \tfrac{1}{2}(c^{-2} + a^{-2}) + \tfrac{1}{2}(c^{-2} - a^{-2})\cos.((\rho\rho') \pm (\rho\rho'')), \qquad (\mathrm{W}^{18})$$

ρ being the radius-vector or velocity and $(\rho\rho')$ $(\rho\rho'')$ being the angles which this radius ρ makes with two constant radii ρ', ρ'', determined by the following cosines of their inclinations to the semiaxes of x y z, or of a b c,

$$\rho'_a = \rho''_a = \sqrt{\frac{b^{-2} - a^{-2}}{c^{-2} - a^{-2}}}, \quad \rho'_b = \rho''_b = 0, \quad \rho'_c = -\rho''_c = \sqrt{\frac{c^{-2} - b^{-2}}{c^{-2} - a^{-2}}}. \qquad (\mathrm{X}^{18})$$

The expression (W^{18}), for the reciprocal of the square of the velocity of a ray, has been assigned by Fresnel, who has also remarked that it gives always two unequal velocities unless the direction ρ of the ray coincide with some one of the four directions $\pm\rho'$, $\pm\rho''$, which are opposite two by two, and situated in the plane a c of the extreme axes of elasticity. Fresnel has shown in like manner that any given normal direction corresponds to two unequal normal velocities, except four particular directions, which we may call $\pm\omega'$, $\pm\omega''$, and which are determined by the following cosines of direction,

$$\omega'_a = -\omega''_a = \sqrt{\frac{a^2 - b^2}{a^2 - c^2}}, \quad \omega'_b = \omega''_b = 0, \quad \omega'_c = \omega''_c = \sqrt{\frac{b^2 - c^2}{a^2 - c^2}}: \qquad (\mathrm{Y}^{18})$$

and in fact it is easy to establish the following expression for the double value of the square of the normal velocity, analogous to the expression (W^{18}),

$$\omega^2 = \tfrac{1}{2}(a^2 + c^2) + \tfrac{1}{2}(a^2 - c^2)\cos.((\omega\omega') \pm (\omega\omega'')), \qquad (\mathrm{Z}^{18})$$

which cannot reduce itself to a single value, unless the sine of $(\omega\omega')$ or of $(\omega\omega'')$ vanishes. Fresnel has given the name of *optic axes* sometimes to the one and sometimes to the other of the two sets of directions (X^{18}) (Y^{18}); but to prevent the confusion which might arise from this double use of a term, we shall, for the present, call the set $\pm\rho'$, $\pm\rho''$, by the longer but more expressive name of the directions or *lines of single ray-velocity*: and similarly we shall call the set $\pm\omega'$ $\pm\omega''$, the directions or *lines of single normal velocity.*

New Properties of Fresnel's *Wave. This Wave has Four Conoidal Cusps, at the Ends of the Lines of Single Ray-Velocity; it has also Four Circles of Contact, of which each is contained on a Touching Plane of Single Normal-Velocity. The Lines of Single Ray-Velocity may therefore be called Cusp-Rays; and the Lines of Single Normal-Velocity may be called Normals of Circular Contact.*

28. The reasonings of the foregoing number suppose that the axes of co-ordinates coincide with the axes of elasticity; but it is easy to extend the result thus obtained, to any other axes of co-ordinates, by the formulæ of transformation which were given in the thirteenth number. We shall content ourselves at present with considering two remarkable transformations of this kind, suggested by the two foregoing sets of lines of single velocity, which conduct to some new properties of Fresnel's wave, and to some new consequences of his theory.

The polar equation (W^{18}) of the wave may be put under the form

$$1 = \tfrac{1}{2}(c^{-2} + a^{-2})\rho^2 + \tfrac{1}{2}(c^{-2} - a^{-2})\rho^2\{r'r'' \pm \sqrt{\rho^2 - r'^2}\sqrt{\rho^2 - r''^2}\}, \tag{A^{19}}$$

if we put for abridgment

$$r' = A'\rho = x\rho'_a + z\rho'_c, \quad r'' = A''\rho = x\rho''_a + z\rho''_c, \tag{B^{19}}$$

so that r', r'', are the projections of the radius-vector ρ on the directions ρ', ρ'', of single ray-velocity; and if we take new rectangular co-ordinates, $x_\prime\ y_\prime\ z_\prime$, such that the plane of $x_\prime\ z_\prime$ is still the plane $a\ c$ of the extreme axes of elasticity, but that the positive semiaxis of $z_\prime$ coincides with the line ρ', we may employ the following formulæ of transformation

$$x = x_\prime\rho'_c + z_\prime\rho'_a, \quad y = y_\prime, \quad z = -x_\prime\rho'_a + z_\prime\rho'_c, \tag{C^{19}}$$

which give

$$\rho^2 = x_\prime^2 + y_\prime^2 + z_\prime^2, \quad r' = z_\prime, \quad r'' = x_\prime \sin.(\rho'\rho'') + z_\prime \cos.(\rho'\rho''), \tag{D^{19}}$$

and change the equation (A^{19}) of the wave to the form

$$\begin{aligned} 1 = b^{-2}z_\prime^2 &+ \tfrac{1}{2}z_\prime x_\prime(c^{-2} - a^{-2})\sin.(\rho'\rho'') + \tfrac{1}{2}(c^{-2} + a^{-2})(x_\prime^2 + y_\prime^2) \\ &\pm \tfrac{1}{2}(c^{-2} - a^{-2})\sqrt{x_\prime^2 + y_\prime^2}\sqrt{(z_\prime \sin.(\rho'\rho'') - x_\prime\cos.(\rho'\rho''))^2 + y'^2}. \end{aligned} \tag{E^{19}}$$

This equation enables us easily to examine the shape of the wave near the end of the radius ρ', that is, near the point having for its new co-ordinates

$$x_\prime = 0, \quad y_\prime = 0, \quad z_\prime = b; \tag{F^{19}}$$

for it takes, near that point, the following approximate form,

$$z_\prime = b - \tfrac{1}{2}b^2\sqrt{c^{-2} - b^{-2}}\,\sqrt{b^{-2} - a^{-2}}\,(x_\prime \pm \sqrt{x_\prime^2 + y_\prime^2}), \tag{G^{19}}$$

which shows that at the point (F^{19}) *the wave has a conoidal cusp*, and is touched not by one determined tangent plane but by *a tangent cone of the second degree*, represented rigorously by the equation (G^{19}). FRESNEL does not appear to have been aware of the existence of this tangent cone to his wave; he seems to have thought that at the end of a radius ρ' of single ray-velocity the wave was touched only by two right lines, contained in the plane of $a\ c$, namely, by the tangents to a certain circle and ellipse, the intersections of the wave with that plane: but it is evident from the foregoing transformation that every other section of the wave, made by a plane containing the radius-vector ρ', is touched, at the end of that radius, by two tangent lines, contained on the cone (G^{19}). It is evident also that there are *four such conoidal cusps*, at the ends of the for lines of single ray-velocity, $\pm\rho'$, $\pm\rho''$. They are determined by the following co-ordinates, when referred to the axes of elasticity,

$$x = \pm c\sqrt{\frac{a^2 - b^2}{a^2 - c^2}}, \quad y = 0, \quad z = \pm a\sqrt{\frac{b^2 - c^2}{a^2 - c^2}}; \tag{H^{19}}$$

and they are the four intersections of FRESNEL'S circle and ellipse, in the plane of a c, which have for their equations in that plane

$$x^2 + z^2 = b^2, \quad a^2x^2 + c^2z^2 = a^2c^2. \tag{I^{19}}$$

Again, if we employ the following new formulæ of transformation,

$$x = x_{\prime\prime}\omega'_c + z_{\prime\prime}\omega'_a, \quad y = y_{\prime\prime}, \quad z = -x_{\prime\prime}\omega'_a + z_{\prime\prime}\omega'_c, \tag{K^{19}}$$

so as to pass to a new system of rectangular co-ordinates such that the plane of $x_{\prime\prime}$ $z_{\prime\prime}$ coincides with the plane of a c, and the positive semiaxis of $z_{\prime\prime}$ with the line ω' of single normal velocity, we find a new transformed equation of the wave, which may be thus written,

$$(x_{\prime\prime}^2 + y_{\prime\prime}^2 + x_{\prime\prime}z_{\prime\prime}b^{-2}\sqrt{a^2 - b^2}\sqrt{b^2 - c^2})^2 = Q(1 - z_{\prime\prime}^2b^{-2}), \tag{L^{19}}$$

if we put for abridgment

$$Q = (a^2 + c^2)\rho^2 + (a^2 - c^2)r'r'' - a^2c^2(1 + z_{\prime\prime}^2b^{-2}); \tag{M^{19}}$$

and hence it is easy to prove that *the plane*

$$z_{\prime\prime} = b, \tag{N^{19}}$$

which is perpendicular to the line ω' at its extremity, touches the wave in the whole extent of a circle; the equation of this circle of contact being, in its own plane,

$$x_{\prime\prime}^2 + y_{\prime\prime}^2 + x_{\prime\prime}b^{-1}\sqrt{a^2 - b^2}\sqrt{b^2 - c^2} = 0. \tag{O^{19}}$$

It is evident that there are *four such circles of plane contact at the ends of the four lines* $\pm\omega'$, $\pm\omega''$, *of single normal-velocity.* They are all equal to each other, and the common magnitude of their diameters is $b^{-1}\sqrt{a^2 - b^2}\sqrt{b^2 - c^2}$. The same conclusions may be drawn from FRESNEL'S equation of the wave in co-ordinates x y z, referred to the axes of elasticity: the equations of the *four planes of circular contact* being, in these co-ordinates,

$$z\sqrt{b^2 - c^2} \pm x\sqrt{a^2 - b^2} = \pm b\sqrt{a^2 - c^2}. \tag{P^{19}}$$

FRESNEL however does not himself appear to have suspected the existence of these circles of contact, nor do they seem to have been since perceived by any other person. We shall find that the circles and cusps, pointed out in the present number, conduct to some remarkable theoretical conclusions respecting the laws of refraction in biaxal crystals.

New Consequences of FRESNEL'S *Principles. It follows from those Principles, that Crystals of sufficient Biaxal Energy ought to exhibit two kinds of Conical Refraction, an External* *and an Internal: a Cusp-Ray giving an External Cone of Rays, and a Normal of Circular Contact being connected with an Internal Cone.*

29. The general formulæ for reflexion or refraction, ordinary or extraordinary, which we have deduced from the nature of the characteristic function V, become simply

$$\Delta\sigma = 0, \quad \Delta\tau = 0, \tag{Q^{19}}$$

when we take for the plane of $x\ y$ the tangent plane to the reflecting or refracting surface; they show therefore that *the components of normal slowness parallel to this tangent plane are not changed*, which is a new and general form for the laws of reflexion and refraction. It is easy to combine this general theorem with FRESNEL'S law of velocity, and so to deduce new consequences from that law with respect to biaxal crystals.

For this deduction, our theorem may be expressed as follows,

$$0 = \Delta\left(a_t \frac{\delta v}{\delta\alpha} + b_t \frac{\delta v}{\delta\beta} + c_t \frac{\delta v}{\delta\gamma}\right), \tag{R^{19}}$$

in which v is the undulatory slowness of a ray considered as a homogeneous function of the first dimension of the cosines $\alpha\ \beta\ \gamma$ of its inclinations to any three rectangular semiaxes $a\ b\ c$, while Δ refers to the changes produced by reflexion or refraction, the unaltered trinomial to which it is prefixed being the component of normal slowness in the direction of any line t on the tangent plane of the reflecting or refracting surface, and $a_t\ b_t\ c_t$ being the cosines of the inclinations of this line to the semiaxes $a\ b\ c$: and in order to combine this theorem with the principles of FRESNEL, we have only to suppose that the rectangular semiaxes $a\ b\ c$ in each medium are the semiaxes of elasticity of that medium, and that the form of the function v is determined as in the twenty-seventh number.

Thus, to calculate the refraction of light on entering from a vacuum into a biaxal crystal $a\ b\ c$ bounded by a plane face F, we may denote by $\alpha_0\ \beta_0\ \gamma_0$ the cosines of the inclinations of the internal or incident ray to two rectangular lines s, t upon the face F, and to the inward normal, and we shall have the two equations following,

$$\left.\begin{aligned} \alpha_0 &= a_s \frac{\delta v}{\delta\alpha} + b_s \frac{\delta v}{\delta\beta} + c_s \frac{\delta v}{\delta\gamma}\ \ (= \sigma a_s + \tau b_s + v c_s), \\ \beta_0 &= a_t \frac{\delta v}{\delta\alpha} + b_t \frac{\delta v}{\delta\beta} + c_t \frac{\delta v}{\delta\gamma}\ \ (= \sigma a_t + \tau b_t + v c_t), \end{aligned}\right\} \tag{S^{19}}$$

which contain the required connexions between $\alpha_0\ \beta_0\ \gamma_0$ and $\alpha\ \beta\ \gamma$, that is, between the external and internal directions. In this manner we find in general two incident rays for one refracted, and two refracted for one incident; because a given system of values of $\alpha\ \beta\ \gamma$, that is, a given direction of the internal ray, corresponds in general to two systems of values of the internal components of normal slowness σ, τ, v, and therefore to two systems of values $\alpha_0\ \beta_0\ \gamma_0$, that is, to two external directions; while, reciprocally, a given system of two linear relations between σ, τ, v, deduced by (S^{19}) from a given external direction, corresponds in general to two directions of the internal ray. But there are two remarkable exceptions, connected with the two sets of lines of single velocity, and with the conoidal cusps and circles of contact on FRESNEL'S wave.

For we have seen that at a conoidal cusp the tangent plane to the wave is indeterminate; it is evident therefore that a *cusp-ray* must correspond to an infinite variety of systems of components of normal slowness σ, τ, υ, within the biaxal crystal, and therefore also to an infinite variety of systems of direction-cosines α_0 β_0 γ_0 of the external ray; so that *this one internal cusp-ray must correspond to an external cone of rays, according to a new theoretical law of light* which may be called EXTERNAL CONICAL REFRACTION.

And again, at a circle of contact, the wave has one common tangent plane for all the points of that circle, and therefore the infinite variety of internal rays which correspond to these different points have all one common wave-normal, which may be called a *normal of circular contact*, and all these internal rays have one common system of components of normal slowness σ τ υ within the crystal, and consequently correspond to one common external ray: so that *this one external ray is connected with an internal cone of rays, according to another new theoretical law of light*, which may be called INTERNAL CONICAL REFRACTION.

To develope, somewhat more fully, these two new consequences from FRESNEL'S principles, let us begin by considering *external conical refraction*: and let us seek the equation of the external cone of rays, corresponding to the internal cusp-ray ρ'. The approximate equation (G^{19}) of the wave, near the end of this cusp-ray, in the transformed co-ordinates $x_{\prime}$ $y_{\prime}$ $z_{\prime}$, gives the following approximate expression for the undulatory slowness v of a near ray, considered as a homogeneous function of the first dimension of the cosines $\alpha_{\prime}$ $\beta_{\prime}$ $\gamma_{\prime}$ of its inclinations to the positive semiaxes of these co-ordinates $x_{\prime}$ $y_{\prime}$ $z_{\prime}$,

$$v = b^{-1}\gamma_{\prime} + r_{\prime}(a_{\prime} \pm \sqrt{\alpha_{\prime}^2 + \beta_{\prime}^2}), \tag{T^{19}}$$

in which

$$r_{\prime} = \tfrac{1}{2} b\sqrt{c^{-2} - b^{-2}}\sqrt{b^{-2} - a^{-2}}; \tag{U^{19}}$$

it gives therefore by our general method, the following components of normal slowness parallel to the same semiaxes of $x_{\prime}$ $y_{\prime}$ $z_{\prime}$,

$$\left.\begin{aligned}
\sigma\rho'_c - \upsilon\rho'_a = \sigma_{\prime} = \frac{\delta v}{\delta\alpha_{\prime}} &= r_{\prime} \pm \frac{r_{\prime}\alpha_{\prime}}{\sqrt{\alpha_{\prime}^2 + \beta_{\prime}^2}},\\
\tau = \tau_{\prime} = \frac{\delta v}{\delta\beta_{\prime}} &= \quad\ \pm \frac{r_{\prime}\beta_{\prime}}{\sqrt{\alpha_{\prime}^2 + \beta_{\prime}^2}},\\
\sigma\rho'_a - \upsilon\rho'_c = \upsilon_{\prime} = \frac{\delta v}{\delta\gamma_{\prime}} &= b^{-1},
\end{aligned}\right\} \tag{V^{19}}$$

the expressions for $\sigma_{\prime}$ $\tau_{\prime}$ becoming indefinitely more accurate as $\alpha_{\prime}$ $\beta_{\prime}$ diminish, that is, as the near internal ray approaches to the cusp-ray ρ', and the expression for $\upsilon_{\prime}$ being rigorous: *the relations between the components of normal slowness σ τ υ of the cusp-ray ρ'* are therefore

$$(\sigma\rho'_c - \upsilon\rho'_a)^2 + \tau^2 = 2r_{\prime}(\sigma\rho'_c - \upsilon\rho'_a), \quad \sigma\rho'_a + \upsilon\rho'_c = b^{-1}, \tag{W^{19}}$$

and *the equation (in α_0 β_0) of the external cone of rays corresponding to the one internal cusp-ray ρ' is to be found by eliminating these three internal components σ τ υ between the two relations* (W^{19}) *and the two equations of refraction* (S^{19}).

For example, if the internal cusp-ray ρ' coincide with the inward normal to the refracting face F of the crystal, we may take, for the semiaxes s, t upon that face, the projection of a, and the semiaxis b of elasticity; and then the equations of refraction (S^{19}) becoming

$$\alpha_0 = \sigma\rho'_c - \upsilon\rho'_a, \beta_0 = \tau, \tag{X^{19}}$$

we have, by (W^{19}), the following polar equation of the external cone of rays

$$\alpha_0^2 + \beta_0^2 = 2r_{\prime}\alpha_0; \tag{Y^{19}}$$

or, in rectangular co-ordinates, an equation of the fourth degree,

$$(x_0^2 + y_0^2)^2 = 4r_{\prime}^2 x_0^2(x_0^2 + y_0^2 + z_0^2). \tag{Z^{19}}$$

This cone is nearly circular in all the known biaxal crystals, because the coefficient $r_{\prime}$ is small, by (U^{19}), when the biaxal energy is weak, that is, when the semiaxes of elasticity a b c are nearly equal to each other: and rigorously the external cone (Z^{19}) meets the concentric sphere of radius unity in a curve contained on a circular cylinder of radius $= r_{\prime}$, one side of this cylinder coinciding with a ray of the cone.

With respect to the *internal conical refraction*, the *equation of the internal cone of rays corresponding to the internal wave-normal ω'*, or *normal of circular contact*, is always, by (N^{19}) (O^{19}),

$$x_{\prime\prime}^2 + y_{\prime\prime}^2 + 2r_{\prime\prime}x_{\prime\prime}z_{\prime\prime} = 0, \text{ if } r_{\prime\prime} = \tfrac{1}{2}b^{-2}\sqrt{a^2 - b^2}\sqrt{b^2 - c^2}, \tag{A^{20}}$$

when referred to the rectangular co-ordinates $x_{\prime\prime}$ $y_{\prime\prime}$ $z_{\prime\prime}$ by the transformation (K^{19}); and in the simpler rectangular co-ordinates x y z which are parallel to the axes of elasticity the equation of this cone is

$$(x\omega'_c - z\omega'_a)^2 + y^2 + 2r_{\prime\prime}(x\omega'_c - z\omega'_a)(x\omega'_a + z\omega'_c) = 0, \tag{B^{20}}$$

in which we may change the co-ordinates x y z to the direction-cosines α β γ of an internal ray of the cone: while the one external ray corresponding is determined by the following direction-cosines

$$\alpha_0 = b^{-1}\omega'_s, \quad \beta_0 = b^{-1}\omega'_t; \tag{C^{20}}$$

or by the ordinary law of proportional sines, since the internal wave-normal of circular contact ω', which is one ray of the internal cone, is connected with the external ray by this ordinary law, if we take as the refracting index of the crystal the reciprocal b^{-1} of the mean semiaxis of elasticity. It is evident hence that if the internal cone emerge at a new plane face, it will *emerge a cylinder*, whether the two faces be parallel or inclined, that is, whether the crystal be a plate or a prism.

Theory of Conical Polarisation. Lines of Vibration. These Lines, on FRESNEL'S *Wave, are the Intersections of Two Series of Concentric and Co-axal Ellipsoids.*

30. A given direction of a wave-normal in a biaxal crystal corresponds in general to two directions of vibration, and therefore to two planes of polarisation, determined by the equations (F^{18}), namely one for each of the two values ω_1^2, ω_2^2 of the square of the normal velocity deduced by (G^{18}) from the given system of ratios of σ, τ, υ; and these two directions of vibration, or the two planes of polarisation, that is, the two normal planes of the wave perpendicular to these vibrations, are perpendicular to each other, since we can easily deduce from (G^{18}) the following relation between ω_1^2, ω_2^2,

$$\frac{\sigma^2}{(\omega_1^2 - a^2)(\omega_2^2 - a^2)} + \frac{\tau^2}{(\omega_1^2 - b^2)(\omega_2^2 - b^2)} + \frac{\upsilon^2}{(\omega_1^2 - c^2)(\omega_2^2 - c^2)} = 0 : \qquad (\mathrm{D}^{20})$$

which general rectangularity of the two vibrations on any one plane wave has been otherwise established by FRESNEL, and is an important result of his theory. But besides this general *double polarisation* connected with the general *double refraction* in biaxal crystals, we may consider two other kinds which may be called *conical polarisation*, connected with the two kinds of *conical refraction*, which were pointed out in the foregoing number.

To examine the law of the conical polarisation connected with the internal conical refraction, and therefore with the planes of circular contact, we may employ the co-ordinates $x_{\prime\prime}$ $y_{\prime\prime}$ $z_{\prime\prime}$ defined by (K^{19}), and thus transform the general equations of polarisation (A^{18}) (F^{18}) into the following equally general,

$$\left.\begin{aligned} \frac{\omega_c'\,\delta x_{\prime\prime} + \omega_a'\,\delta z_{\prime\prime}}{\omega_c'\sigma_{\prime\prime} + \omega_a'\upsilon_{\prime\prime}}(\omega^2 - a^2) = \frac{\delta y_{\prime\prime}}{\tau_{\prime\prime}}(\omega^2 - b^2) = \frac{-\omega_a'\,\delta x_{\prime\prime} + \omega_c'\,\delta z_{\prime\prime}}{-\omega_a'\sigma_{\prime\prime} + \omega_c'\upsilon_{\prime\prime}}(\omega^2 - c^2), \\ \sigma_{\prime\prime}\,\delta x_{\prime\prime} + \tau_{\prime\prime}\,\delta y_{\prime\prime} + \upsilon_{\prime\prime}\,\delta z_{\prime\prime} = 0; \end{aligned}\right\} \qquad (\mathrm{E}^{20})$$

which give, for the projection of a vibration on the plane $x_{\prime\prime}y_{\prime\prime}$ of single normal velocity, the rigorous formula

$$\frac{\delta y_{\prime\prime}}{\delta x_{\prime\prime}} = \frac{(\omega^2 - a^2)(\omega^2 - c^2)}{\omega^2 - b^2} \, \frac{\tau_{\prime\prime}}{\upsilon_{\prime\prime}\sqrt{a^2 - b^2}\sqrt{b^2 - c^2} + \sigma_{\prime\prime}(\omega^2 + b^2 - a^2 - c^2)}, \qquad (\mathrm{F}^{20})$$

and for any plane wave slightly inclined to this plane of $x_{\prime\prime}y_{\prime\prime}$ the following approximate relation between the components of normal slowness,

$$\upsilon_{\prime\prime} = b^{-1} + r_{\prime\prime}(\sigma_{\prime\prime} \pm \sqrt{\sigma_{\prime\prime}^2 + \tau_{\prime\prime}^2}), \qquad (\mathrm{G}^{20})$$

retaining the meaning (A^{20}) of $r_{\prime\prime}$; and if we attend to the general connexions, established in this Supplement, between the direction-cosines of a ray and the components of normal slowness of a wave, we easily deduce from (G^{20}), by differentiation, the following other relations,

$$\frac{\alpha_{\prime\prime}}{\gamma_{\prime\prime}} = -\frac{\delta\upsilon_{\prime\prime}}{\delta\sigma_{\prime\prime}} = -r_{\prime\prime}\left(1 \pm \frac{\sigma_{\prime\prime}}{\sqrt{\sigma_{\prime\prime}^2 + \tau_{\prime\prime}^2}}\right), \quad \frac{\beta_{\prime\prime}}{\gamma_{\prime\prime}} = -\frac{\delta\upsilon_{\prime\prime}}{\delta\tau_{\prime\prime}} = \frac{\mp r_{\prime\prime}\tau_{\prime\prime}}{\sqrt{\sigma_{\prime\prime}^2 + \tau_{\prime\prime}^2}}; \qquad (\mathrm{H}^{20})$$

and finally for the vibrations of a near wave

$$\frac{\delta y_{\prime\prime}}{\delta x_{\prime\prime}} = \frac{\tau_{\prime\prime}}{\sigma_{\prime\prime} \pm \sqrt{\sigma_{\prime\prime}^2 + \tau_{\prime\prime}^2}} = \frac{\beta_{\prime\prime}}{\alpha_{\prime\prime}}. \tag{I^{20}}$$

This formula contains the theory of the conical polarisation connected with internal conical refraction. It shows that *the vibrations at the circle of contact on* FRESNEL'S *wave are in the chords of that circle drawn from the extremity of the normal ω' of single velocity*; and therefore that *the corresponding planes of polarisation all pass through another parallel normal at the opposite point of the circle.* The plane of polarisation, therefore, in passing from one position to another, *revolves only half as rapidly* as the revolving radius, so that the angle between any two planes of polarisation is only *half* the angle between the two corresponding radii of this circle on FRESNEL'S wave. And if we suppose that the direction of the external incident ray coincides with the wave-normal ω', and therefore also with the normal to the refracting face of the crystal, then the small internal components of normal slowness $\sigma_{\prime\prime}$ $\tau_{\prime\prime}$, parallel to this refracting face, are equal (by our general theorem of refraction) to the small external direction-cosines α_0 β_0 of the inclinations of a near incident ray to the semiaxes of $x_{\prime\prime}$ and $y_{\prime\prime}$; from which it follows, by (I^{20}) that *the plane of external incidence containing this near incident ray revolves twice as rapidly as the corresponding plane of refraction.*

For the other kind of conical polarisation, connected with the external conical refraction, and therefore with the conoidal cusps on FRESNEL'S wave, we find by a similar process,

$$\frac{\delta y_{\prime}}{\delta x_{\prime}} = \frac{\tau_{\prime}}{\sigma_{\prime}} = \frac{\beta_{\prime}}{\alpha_{\prime} \pm \sqrt{\alpha_{\prime}^2 + \beta_{\prime}^2}}, \tag{K^{20}}$$

and

$$\delta z_{\prime} = -2br_{\prime}\,\delta x_{\prime}, \tag{L^{20}}$$

$r_{\prime}$ having the meaning (U^{19}). The formula (K^{20}) shows that the normal plane to the wave, containing any vibration near the cusp, contains either the cusp-ray itself, or a line parallel to this ray; so that the direction of any near vibration coincides with or is parallel to the projection of the cusp-ray on the corresponding tangent plane of the wave, or of the cone which touches it at the cusp: and the formula (L^{20}) shows that all these near vibrations are parallel to one common plane, which is easily seen to be perpendicular to the plane of a c, and to contain the tangent at the cusp to the elliptic section (I^{19}) of the wave, made by this latter plane; so that *all the planes of polarisation near the cusp, contain, or are parallel to, the normal of this elliptic section.* And the direction of any near vibration on the wave, or on its tangent cone, may be obtained by cutting the corresponding tangent plane of this wave or cone by a plane perpendicular to this elliptic normal.

If the cusp-ray be incident perpendicularly on a refracting face of the crystal, then the internal components $\sigma_{\prime}$ $\tau_{\prime}$ are equal to the direction-cosines α_0 β_0 of the corresponding ray of the emerging external cone; and therefore, by (K^{20}), the plane of refraction of this external ray contains the internal vibration, and therefore also, by FRESNEL'S principles, the external vibration corresponding: so that, *in the external conical polarisation, produced by the perpendicular internal incidence of a cusp-ray, the plane of polarisation of an external* *ray is perpendicular to its plane of refraction; and therefore revolves about half as rapidly as the plane containing this emergent ray and passing through the approximate axis of the nearly circular emergent cone, when the biaxal energy is small.* We see also, by (K^{20}), that the plane containing the cusp-ray and containing or parallel to a near internal ray, revolves with double the rapidity of the plane containing the cusp-ray and parallel to the near wave-normal; and therefore, in the case of perpendicular incidence of the cusp-ray, the plane of incidence of a near internal ray revolves with double the rapidity of the plane of external refraction, which, as we have seen, contains here the external vibrations.

In general, the equations of polarisation (F^{18}), which we have deduced from FRESNEL'S principles, conduct, by (I^{18}) (L^{18}), to the following simple formula

$$a^2\alpha\,\delta x + b^2\beta\,\delta y + c^2\gamma\,\delta z = 0, \tag{M^{20}}$$

δx, δy, δz being still the components of displacement parallel to the semiaxes a, b, c, and α, β, γ being still the cosines of the inclinations of the ray to the same semiaxes of elasticity: and this formula (M^{20}), when combined with the equation of transversal vibrations,

$$\delta V = 0, \quad \text{ or, } \quad \sigma\,\delta x + \tau\,\delta y + \upsilon\,\delta z = 0, \tag{A^{18}}$$

determines easily the direction of vibration for any given direction and velocity of a ray, that is, for any point of FRESNEL'S curved wave propagated from a luminous origin within a biaxal crystal. And we easily see that *on any wave in a biaxal crystal*, whether propagated from within or from without, the differential equation (M^{20}) determines *a series of lines of vibration*, having the property that at any point of such a line the vibration is in the direction of the line itself. To find these lines on FRESNEL'S wave (O^{18}), we may change α β γ to x y z in the differential equation (M^{20}), and we then find, by integration,

$$a^2x^2 + b^2y^2 + c^2z^2 = \varepsilon^4, \tag{N^{20}}$$

ε being an arbitrary constant; and since this integral, when combined with the equation (O^{18}) of the wave itself, gives

$$(a^2+\varepsilon^4)x^2 + (b^2+\varepsilon^4)y^2 + (c^2+\varepsilon^4)z^2 = (a^2+b^2+c^2)\varepsilon^4 - a^2b^2c^2, \tag{O^{20}}$$

we see that *the lines of vibration on* FRESNEL'S *wave, propagated from a point in a biaxal crystal, are the intersections of two series* (N^{20}) (O^{20}) *of concentric and co-axal ellipsoids.*

By this general integration, extending to the whole wave, or by integrating the approximate equations for vibrations near the conoidal cusps and circles of contact, obtained from (K^{20}) (I^{20}) by changing the direction-cosines of a ray to the proportional co-ordinates of the wave, we find that near a cusp the lines of vibration coincide nearly with small parabolic arcs on the tangent cone of the wave, in planes perpendicular to the elliptic normal already mentioned; and that in crossing a circle of contact the course of each line of vibration is directed towards that point of the circle which is the end of the corresponding wave-normal of single velocity, that is, towards the foot of the perpendicular let fall from the centre of the wave on the plane of circular contact.

In any Uniform Medium, the Curved Wave propagated from a point is connected with a certain other surface, which may be called the Surface of Components, by relations discovered by M. Cauchy, *and by some new relations connected with a General Theorem of Reciprocity. This new Theorem of Reciprocity gives a new construction for the Wave, in any Undulatory Theory of Light: and it connects the Cusps and Circles of Contact on* Fresnel's *Wave, with Circles and Cusps of the same kind on the Surface of Components.*

31. The theory of the wave propagated from a point in any uniform medium may be much illustrated by comparing this wave with a certain other surface which appears to have been first discovered by M. Cauchy, who has pointed out some of its properties in the *Livraison* already referred to. In that *Livraison*, M. Cauchy has treated of the propagation of plane waves in a system of mutually attracting or repelling particles; and has been conducted to a relation between the normal velocity of propagation, which he calls s, and the cosines of its inclinations to the positive semiaxes of x, y, z, which cosines he denotes by a, b, c. The relation thus found being expressed by equating to zero a certain homogeneous function (of the sixth dimension) of s, a, b, c, it has suggested to M. Cauchy the consideration of s as a homogeneous function of the first dimension of the cosines a, b, c, whereas we have preferred to treat the normal velocity (denoted in this Supplement by ω) as a homogeneous function of its cosines of direction of the dimension zero; a difference in method which makes no real difference in the results, because the relation existing between the cosines (namely, that the sum of their squares is unity,) permits us to transform in an infinite variety of ways any equation into which they enter. M. Cauchy deduces from his view of the relation between the normal velocity and cosines of normal direction, the following equations between the time t and the co-ordinates x y z, of a ray from the origin of co-ordinates,

$$\frac{x}{t} = \frac{ds}{da}, \quad \frac{y}{t} = \frac{ds}{db}, \quad \frac{z}{t} = \frac{ds}{dc},$$

which were alluded to in the twenty-sixth number of the present Supplement, as substantially equivalent to our equations (D^{18}). He deduces also an equation of the form

$$F\left(\frac{a}{s}, \frac{b}{s}, \frac{c}{s}\right) = 0,$$

which he constructs by a surface having $\frac{a}{s}$, $\frac{b}{s}$, $\frac{c}{s}$, for its co-ordinates. Our methods suggest immediately the same surface, as the construction of the same equation under the form

$$\Omega(\sigma, \tau, \upsilon) = 0,$$

which has been so frequently employed in this Supplement; and from the optical meanings that we have pointed out for the co-ordinates σ, τ, υ, of this surface $\Omega = 0$, we shall call it the surface of components of normal slowness, or simply *the surface of components.* M. Cauchy shows that this surface is connected with the curved wave propagated from the origin of co-ordinates in the unit of time, (which we have called the *unit-wave* and may denote by the equation

$$V = 1,)$$

by two remarkable relations, which can easily be deduced from our formulæ, and may be thus enunciated: first, *the sum of the products of their corresponding co-ordinates*, or, in other words, *the product of any two corresponding radii multiplied by the cosine of the included angle, is unity*; and secondly, *the wave is the enveloppe of the planes which cut perpendicularly the radii of the surface of components at distances from the centre equal to the reciprocals of those radii.*

To these two relations, discovered by M. Cauchy, we may add a third, not less remarkable, which he does not seem to have perceived: namely, that *the surface of components is the enveloppe of the planes which cut perpendicularly the radii of the wave at distances from its centre equal to the reciprocals of those radii*, that is, equal to the slowness of the rays. For it is a general theorem of reciprocity between surfaces, which can easily be deduced from the evident coexistence of the three equations

$$\left.\begin{aligned} xx' + yy' + zz' &= 1, \\ x\,\delta x' + y\,\delta y' + z\,\delta z' &= 0, \\ x'\,\delta x + y'\,\delta y + z'\,\delta z &= 0, \end{aligned}\right\} \qquad (\mathrm{P}^{20})$$

that *if one surface B be deduced from another A by drawing radii vectores to the latter from an arbitrary origin O, and altering the lengths of these radii to their reciprocals without changing their directions, and seeking the enveloppe B of the planes perpendicular at the extremities to these altered radii of A, then reciprocally, the surface A may be deduced from B by a repetition of the same construction*, employing the same origin O, and the same arbitrary unit of length. For example, if the surface A be formed by the revolution of an ellipse about its major axis, and if we place the arbitrary origin O at one focus of this ellipsoid A, and take the arbitrary unit equal to the semiaxis minor, the enveloped surface B will be a sphere, having its diameter equal to the axis major of the ellipsoid, and its centre on that axis major, the interval between the centres of the two surfaces being bisected by the origin O; and if from this excentric origin we draw radii to the sphere B, and change these unequal radii to their reciprocals, and draw perpendicular planes at the extremities of these new radii, the enveloppe of the planes so drawn will be the ellipsoid A. Another particular case of this general theory of *reciprocal surfaces*, namely the case of two concentric and co-axal ellipsoids, referred to their centre as origin, and having the semiaxes of one equal to the reciprocals of those of the other, has been perceived by Mr. Mac Cullagh, and elegantly proved by him, in the Second Part of the Sixteenth Volume of the *Transactions of the Royal Irish Academy.*

This general theorem of reciprocity, when applied to the unit-wave and surface of components, gives *a new construction for the unit-wave in any uniform medium, and for any law of velocity*: namely, that *the wave is the locus of the points obtained by letting fall perpendiculars from the centre on the tangent planes of the surface of components, and then altering the lengths of these perpendiculars to their reciprocals, without altering their directions.*

It follows also from this general theory of *reciprocal surfaces*, that *a conoidal cusp on any surface A corresponds in general to a curve of plane contact on the reciprocal surface B*, and reciprocally; and accordingly the cusps and circles on FRESNEL'S wave are connected with circles and cusps on the corresponding surface of components, which latter surface is indeed deducible from the former by merely changing the semiaxes of elasticity a b c to their reciprocals. And it was in fact by this general theorem that I was led to discover the four circles of contact on FRESNEL'S wave, by concluding that this wave must touch four planes in curves instead of points of contact, as soon as I had perceived the existence of four conoidal cusps on the surface of components, by obtaining (in some investigations respecting the aberrations of biaxal lenses) the formula (G^{20}), which is the approximate equation of such a cusp. I easily found also that there were *only four* such cusps on each of the two reciprocal surfaces, and therefore concluded that there were *only four* curves of plane contact on each. I may mention that though I have taken care to attribute to M. CAUCHY the discovery of the surface of components, yet before I met the *Exercices de Mathématiques*, I was familiar, in my own investigations, with the existence and with the foregoing properties of this surface: it is indeed immediately suggested by the first principles of my view of optics, since it constructs the fundamental partial differential equation

$$\Omega\left(\frac{\delta V}{\delta x}, \frac{\delta V}{\delta y}, \frac{\delta V}{\delta z}\right) = 0$$

which my characteristic function V must satisfy in a final uniform medium.

The surface of components possesses many other interesting properties, for example the following, that in a final uniform medium any two conjugate planes of vergency (E^{15}) are perpendicular to two conjugate tangents to it: which is analogous to the less simple relations considered in the twenty-first number. But the length to which this Supplement has extended, confines me here to remarking, that the general equations of reflexion or refraction,

$$\Delta\sigma = 0, \quad \Delta\tau = 0, \qquad (Q^{19})$$

may be thus enunciated; *the corresponding points* (σ, τ, υ, and $\sigma+\Delta\sigma$, $\tau+\Delta\tau$, $\upsilon+\Delta\upsilon$) *upon the surface or surfaces of components* ($0 = \Omega$, $0 = \Omega + \Delta\Omega$,) *before and after any reflexion or refraction, ordinary or extraordinary, are situated on one common perpendicular to the plane which touches the reflecting or refracting surface at the point of reflexion or refraction*; a new geometrical relation, which gives a new and general construction to determine a reflected or refracted ray, simpler in many cases than the construction proposed by HUYGHENS.