ON ADDITIONAL APPLICATIONS OF QUATERNIONS TO SURFACES OF THE SECOND ORDER

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Sir W. Rowan Hamilton gave an account of some additional applications of Quaternions to Surfaces of the Second Order.

In the Abstract printed as part of the Proceedings of the Academy for July 20, 1846, the following equation of the ellipsoid (there numbered 44),

$$\mathrm{T}(\iota\rho + \rho\kappa) = \kappa^2 - \iota^2, \tag{1}$$

was given, as a transformation of this other equation of the same surface (there marked 35):

$$\mathrm{T}(\alpha\rho + \rho\alpha + \beta\rho - \rho\beta) = 1; \tag{2}$$

which was itself deduced by transforming, according to the rules of quaternions, the formula

$$(\alpha\rho + \rho\alpha)^2 - (\beta\rho - \rho\beta)^2 = 1; \tag{3}$$

this last quaternion form of the equation of the ellipsoid having been previously exhibited to the Academy, at its meeting of December 8, 1845. (See the equation numbered 21, in the Proceedings of that date.) The symbols α, β, denote two constant vectors; the symbols ι, κ, denote two other constant vectors, connected with them by the relations

$$\alpha + \beta = \frac{\iota}{\iota^2 - \kappa^2}, \quad \alpha - \beta = \frac{\kappa}{\iota^2 - \kappa^2}, \tag{4}$$

where $\iota^2 - \kappa^2$ is a negative scalar; and ρ denotes a variable vector, drawn from the centre to the surface of the ellipsoid: while T is the characteristic of the operation of taking the *tensor* of a quaternion.

If a new variable vector ν be defined, as a function of the three vectors ι, κ, ρ, by the equation

$$(\kappa^2 - \iota^2)^2\nu = (\kappa^2 + \iota^2)\rho + \iota\rho\kappa + \kappa\rho\iota, \tag{5}$$

it results from the general rules of this calculus that this new vector ν will satisfy each of the two following equations:

$$\mathrm{S}\,.\,\nu\rho = 1; \quad \mathrm{S}\,.\,\nu\,\mathrm{d}\rho = 0; \tag{6}$$

which give also these two other equations, of the same kind with them, and differing only by the interchange of the two symbols ρ and ν:

$$\mathrm{S} \,.\, \rho\nu = 1; \quad \mathrm{S} \,.\, \rho \,\mathrm{d}\nu = 0; \tag{7}$$

where d is the characteristic of differentiation, and S is that of the operation of taking the scalar part of a quaternion. The equations (6) shew that ν is the vector, of which the reciprocal ν^{-1} represents in length and in direction the perpendicular let fall from the common origin of the variable vectors here considered on the plane which touches at the extremity of the vector ρ the locus of that variable extremity; so that ν^{-1} is here a symbol for the perpendicular let fall from the centre of the ellipsoid on the tangent plane to that surface: and ν itself denotes, in length and in direction, the reciprocal of that perpendicular, so that it may be called the *vector of proximity* of the tangent plane, or of the element of the surface of the ellipsoid, to the centre regarded as an origin. Accordingly, the equation here marked (5) was given in the Abstract of July, 1846 (where it was numbered 45), as a formula for determining what was there also called the vector of proximity of the tangent plane of the ellipsoid. It may now be seen that the symbolical connexion between the two equations above marked (6), and the two other equations lately numbered (7), corresponds to, and expresses, in this Calculus, under what may be regarded as a strikingly simple form, the known connexion of *reciprocity* between any two surfaces, of which one is the locus of the extremities of straight lines drawn from any fixed point, so as to be in their directions perpendicular to the tangent planes of the other surface, and in their lengths inversely proportional to those perpendiculars: from the perception of which *general* relation of reciprocity between surfaces, exemplified previously for the case of two reciprocal ellipsoids by that great geometrical genius (Professor Mac Cullagh), whose recent and untimely loss we all so deeply deplore, the author of the present communication was led to announce to the Academy, in October, 1832, the existence of certain *circles of contact* on Fresnel's wave, which he saw to be a necessary consequence of the existence of certain *conical cusps* on another and reciprocal surface. A very elegant *geometrical* proof of the same general theorem of reciprocity was given afterwards, in the Transactions* of this Academy by Professor Mac Cullagh himself.

As respects the reciprocal ellipsoid, of which the vector ν, in the equation lately marked (5), denotes a semidiameter, it may be mentioned here that, with the same significations of the symbols, the following equation holds good:

$$(2\beta\mathrm{S} \,.\, \alpha\beta)^2 = (\beta\mathrm{S} \,.\, \beta\nu)^2 + (\mathrm{V} \,.\, \beta\mathrm{V} \,.\, \alpha\nu)^2; \tag{8}$$

with equations for other central surfaces of the second order, regarded as reciprocals of central surfaces, which differ only in the signs of their terms from this equation (8). The author proposes, in a future continuation of the present communication, to illustrate this new form, as regards the processes of obtaining and of interpreting it. Meanwhile he desires to submit to the notice of the Academy the following construction, for generating a system of two

* See the beautiful paper entitled, "Geometrical Propositions applied to the Wave Theory of Light. By James Mac Cullagh, F. T. C. D." Read June 24, 1833. Transactions of the Royal Irish Academy, vol. xvii.

reciprocal ellipsoids, by means of a moving sphere, to which his own methods have conducted him, although it may turn out to have been already otherwise discovered. Let then a sphere of constant magnitude, with centre E, move so that it always intersects two fixed and mutually intersecting straight lines, AB, AB', in four points, L, M, L', M', of which L and M are on AB and L' and M' are on AB'; and let one diagonal LM', of the inscribed quadrilateral $LMM'L'$, be constantly parallel to a third fixed line AC, which will oblige the other diagonal ML' of the same quadrilateral to move parallel to a fourth fixed line AC'. Let N be the point in which the diagonals intersect, and draw AF equal and parallel to EN; so that $AENF$ is a parallelogram: then *the locus of the centre E of the moving sphere is one ellipsoid, and the locus of the opposite corner F of the parallelogram is another ellipsoid reciprocal thereto.* These two ellipsoids have a common centre, namely, the point A; and a common mean axis, which is equal to the diameter of the moving sphere. Two sides, AE, AF, of the parallelogram $AENF$, are thus two semidiameters, which may be regarded as reciprocal to each other, one of the one ellipsoid, and the other of the other. It is, however to be observed, that they fall at opposite sides of the principal plane, containing the four fixed lines, and that, therefore, it may be proper to call them more fully *opposite reciprocal semidiameters*; and to call the points E and F, in which they terminate, *opposite reciprocal points.* The two other sides EN, FN, of the same varying parallelogram, are the normals to the two ellipsoids, meeting each other in the point N, upon the same principal plane. In that plane, the two former fixed lines, AB, AB', are the axes of the two cylinders of revolution which are circumscribed about the first ellipsoid; and the two latter fixed lines, AC, AC', are the two cyclic normals of the same first ellipsoid: while the diagonals LM', ML', of the inscribed quadrilateral in the construction, are the axes of the two circles on the surface of that ellipsoid, which circles pass through the point E, that is through the centre of the moving sphere, and which are also contained upon the surface of another sphere, having its centre at the point N: all which is easily adapted, by suitable interchanges, to the other or reciprocal ellipsoid, and flows with great facility from the quaternion equations above given.

It may not be out of place to mention, on this occasion, although for the present without its demonstration, another simple geometrical construction connected with a surface of the second order, and derived from the same calculus of quaternions. This construction is adapted to determine the cone of revolution which osculates, along a given side, to a cone of the second degree; but it will perhaps be most easily understood by considering it as serving to assign the interior pole of the small circle on a sphere, which osculates at a given point T, to a given spherical conic. Let the given cyclic arcs be AC, AC', extending from one of the two points A of their own mutual intersection to the tangent arc CTC', which is well known to be bisected at the point of contact T. On the normal arc NTP, drawn through that given point T, let fall a perpendicular arc AN; draw NC, or NC', and erect CP or $C'P$, perpendicular thereto, and meeting the normal arc in P: the point P, thus determined, will be the pole, or spherical centre of curvature, which was required.