ON QUATERNIONS AND THE ROTATION OF A SOLID BODY

By

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Sir William Rowan Hamilton gave an account of some applications of Quaternions to questions connected with the Rotation of a Solid Body.

I. It was shown to the Academy in 1845, among other applications of the Calculus of Quaternions to the fundamental problems of Mechanics, that the composition of statical couples, of the kind considered by Poinsot, as well as that of ordinary forces, admits of being expressed with great facility and simplicity by the general methods of this Calculus. Thus, the general conditions of the equilibrium of a rigid system are included in the following formula, which will be found numbered as equation (20) of the abstract of the Author's communication of December 8, 1845, in the Proceedings of the Academy for that date:

$$\sum . \, \alpha\beta = -c. \tag{1}$$

In the formula thus cited, α is the *vector of application* of a force denoted by the other vector β; and the scalar sysmbol, $-c$, which is equated to the sum $\alpha\beta + \alpha'\beta' + \cdots$ of all the quaternion products $\alpha\beta, \alpha'\beta', \ldots$ of all such pairs of vectors, or directed lines α and β, is, in the case of equilibrium, independent of the position of the point from which all the vectors $\alpha, \alpha', \ldots$ are drawn, as from a common origin, to the points of application of the various forces, $\beta, \beta', \ldots$. This requires that the two following conditions should be separately satisfied:

$$\sum \beta = 0; \quad \sum V . \, \alpha\beta = 0; \tag{2}$$

which accordingly coincide with the two equations marked (18) of the abstract just referred to. The former of these two equations, $\sum \beta = 0$, expresses that the applied *forces* would balance each other, if they were all transported, without any changes in their intensities or directions, so as to act at any one common point, such as the origin of the vectors α; and the latter equation, $\sum V . \alpha\beta = 0$, expresses that all the *couples* arising from such transport of the forces, or from the introduction of a system of new and opposite forces, $-\beta$, all acting at the same common origin, would also balance each other: the *axis* of any one such couple being denoted, in magnitude and in direction, by a symbol of the form $V . \alpha\beta$. When either of these two vector-sums, $\sum \beta$, $\sum V . \alpha\beta$, is different from zero, the system cannot be in equilibrium, at least if there be no fixed point nor axis; and in this case, the quaternion quotient which is obtained, by dividing the latter of these two vector-sums by the former, has a remarkable and simple signification. For, if this division be effected by the general rules of this calculus, in such a manner as to give a quotient expressed under the original and standard form of a *quaternion*, as assigned by Sir William R. Hamilton in his communication of the 13th of November, 1843; that is to say, if the quotient of the two vectors lately mentioned be reduced by those general rules to the fundamental *quadrinomial form*,

$$\frac{\sum V \,.\, \alpha\beta}{\sum \beta} = w + ix + jy + kz, \tag{3}$$

where i, j, k are the Author's *three co-ordinate imaginaries*, or *rectangular vector-units*, namely, symbols satisfying the equations,

$$i^2 = j^2 = k^2 = ijk = -1, \tag{4}$$

which have already been often adduced and exemplified by him, in connexion with other geometrical and physical researches; then the *four constituent numbers*, w, x, y, z, of this quaternion (3), will have, in the present question, the meanings which we are about to state. The algebraically real or *scalar* part of the quaternion (3), namely, the number

$$w = S(\textstyle\sum V \,.\, \alpha\beta \div \sum \beta), \tag{5}$$

which is independent of the imaginary or symbolic coefficients i, j, k, will denote the (real) quotient which might be otherwise obtained by *dividing the moment of the principal resultant couple by the intensity of the resultant force*; with the known direction of which force the axis of this *principal* (and known) couple coincides, being the line which is known by the name of the *central axis* of the system. And the three other numerical constituents of the same quaternion (3), namely, the three real numbers x, y, z, which are multiplied respectively by those symbolic coefficients i, j, k, in the algebraically imaginary or *vector* part of that quaternion, namely, in the part

$$ix + jy + kz = V(\textstyle\sum V \,.\, \alpha\beta \div \sum \beta), \tag{6}$$

are the three real and rectangular *co-ordinates of the foot of the perpendicular let fall from the assumed origin* (of vectors or of co-ordinates) *on the central axis of the system.* These co-ordinates vanish, if the origin be taken on that axis; and then the direction of the resultant force coincides with that of the axis of the resultant couple: a coincidence of which the condition may accordingly be expressed, in the notation of this Calculus, by the formula

$$0 = V(\textstyle\sum V \,.\, \alpha\beta \div \sum \beta); \tag{7}$$

whereas the second member of this formula (7) is in general a vector-symbol, which denotes, in length and in direction, the perpendicular let fall as above. In the case where it is possible to reduce the system of forces to a single resultant force, unaccompanied by any couple, the scalar part of the same quaternion (3) vanishes; so that we may write for this case the equation,

$$0 = S(\textstyle\sum V \,.\, \alpha\beta \div \sum \beta); \tag{8}$$

which agrees with the equation (19) of the abstract of December, 1845, and in which the second member is in general a scalar symbol, denoted lately by w, and having the signification already assigned. When the resultant force vanishes, without the resultant couple vanishing, then the denominator or divisor $\sum\beta$ becomes null, in the fraction or quotient (3), while the numerator or dividend, $\sum V\,.\,\alpha\beta$, continues different from zero; and when both force and couple vanish, we fall back on the equations (18) of the former abstract just cited, or on those marked (2) in the present communication, as the conditions of equilibrium of a free but rigid system. Finally, the scalar symbol

$$c = -\sum S\,.\,\alpha\beta, \tag{9}$$

which enters with its sign changed into the second member of the formula (1), and which, when the resultant $\sum\beta$ of the forces β vanishes, receives a value independent of the assumed origin of the vectors α, has also a simple signification; for (according to a remark which was made on a former occasion), there appears to be a propriety in regarding this scalar symbol c, or the negative of the sum of the scalar parts of all the quaternion products of the form $\alpha\beta$, as an expression which denotes the total *tension* of the system. In the foregoing formulæ the letters S and V are used as characteristics of the operations of taking respectively the *scalar* and the *vector*, considered as the two *parts* of any quaternion expression; which parts may still be sometimes called the (algebraically) *real* and (algebraically) *imaginary* parts of that expression, but of which *both* are *always*, in this theory, entirely and easily *interpretable*: and in like manner, in the remainder of this Abstract, the letters T and U shall indicate, where they occur, the operations of taking separately the *tensor* and the *versor*, regarded as the two principal *factors* of any such quaternion.

II. To apply to problems of *dynamics* the foregoing *statical* formulæ, we have only to introduce, in conformity with a well-known principle of mechanics, the consideration of the equilibrium which must subsist between the forces lost and gained. That is, we are to substitute for the symbol β, in the equations (1) or (2), the expression

$$\beta = m\left(\phi - \frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2}\right); \tag{10}$$

where m denotes the mass of that part or element of the system which, at the time t, has α for its vector of position, and consequently $\dfrac{\mathrm{d}^2\alpha}{\mathrm{d}t^2}$ for its vector of acceleration; while the new vector-symbol ϕ denotes the accelerating force, or $m\phi$ denotes the moving force applied, direction as well as intensity being attended to. Thus, instead of the two statical equations (2), we have now the two following dynamical equations, for the motion of a free but rigid system:

$$\sum.\, m\frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2} = \sum.\, m\phi; \tag{11}$$

$$\sum.\, mV\,.\,\alpha\frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2} = \sum.\, mV\,.\,\alpha\phi; \tag{12}$$

of which the former contains the law of motion of the centre of gravity, and the latter contains the law of the description of areas. If the rigid system have one point fixed, we may place at this point the origin of the vectors α; and in this case the equation (11) disapears from the statement of the question, but the equation (12) still remains: while the condition that the various points of the system are to preserve unaltered their distances from each other, and from the fixed point, is expressed by the formula

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = V \,.\, \iota\alpha, \tag{13}$$

where the vector-symbol ι denotes a straight line drawn in the direction of the axis of momentary rotation, and having a length which represents the angular velocity of the system; so that this vector ι is generally a function of the time t, but is always, at any one instant, the same for all the points of the body, or of the rigid system here considered. The equation (12) thus gives, by an immediate integration, the following expression for the law of areas:

$$\textstyle\sum \,.\, m\alpha V \,.\, \iota\alpha = \gamma + \sum \,.\, mV \displaystyle\int \alpha\phi \,\mathrm{d}t; \tag{14}$$

where γ is a constant vector; and if we operate on the same equation (12) by the characteristic $2S \int \iota \,\mathrm{d}t$, we obtain an expression for the law of living forces, under the form:

$$\textstyle\sum \,.\, m(V \,.\, \iota\alpha)^2 = -h^2 + 2 \displaystyle\sum .mS \int \iota\alpha\phi \,\mathrm{d}t; \tag{15}$$

where h is a constant scalar. The integrals with respect to the time may be conceived to begin with $t = 0$; and then the vector γ will represent the *axis of the primitive couple*, or of the couple resulting from all the moving forces due to the initial velocities of the various points of the body; and the scalar h will represent the *square root of the primitive living force* of the system, or the square root of the sum of all the living forces obtained by multiplying each mass into the square of its own initial velocity. Again, the equation (13) gives by differentiation,

$$\frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2} = V \,.\, \iota\frac{\mathrm{d}\alpha}{\mathrm{d}t} + V \,.\, \frac{\mathrm{d}\iota}{\mathrm{d}t}\alpha = \iota V \,.\, \iota\alpha - V \,.\, \alpha\frac{\mathrm{d}\iota}{\mathrm{d}t}; \tag{16}$$

and for any two vectors α and ι, we have, by the general rules of this Calculus, the transformations,

$$V \,.\, \alpha(\iota V \,.\, \iota\alpha) = V \,.\, \iota(\alpha V \,.\, \iota\alpha) = \tfrac{1}{2}V \,.\, (\iota\alpha)^2 = S \,.\, \iota\alpha \,.\, V \,.\, \iota\alpha = \tfrac{1}{2}V \,.\, \iota(\alpha\iota\alpha) = -\tfrac{1}{2}V \,.\, \alpha(\iota\alpha\iota); \tag{17}$$

therefore, by (12) and (14),

$$\textstyle\sum \,.\, m\alpha V \,.\, \alpha\frac{\mathrm{d}\iota}{\mathrm{d}t} + \sum \,.\, mV \,.\, \alpha\phi = V \,.\, \iota\sum \,.\, m\alpha V \,.\, \iota\alpha = V \,.\, \iota\gamma + \sum \,.\, mV \,.\, \iota V \displaystyle\int \alpha\phi \,\mathrm{d}t. \tag{18}$$

Hence also the *time* t, elapsed between any two successive stages of the rotation of the body, may in various way be expressed by a definite integral; we may, for example, write generally,

$$t = \int \frac{2\sum .m\alpha V \,.\, \alpha \,\mathrm{d}\iota}{\sum V \,.\, m((\iota\alpha)^2 + 2\phi\alpha)}; \tag{19}$$

the scalar element dt of this integral being thus expressed as the quotient of a vector element, divided by another vector; before finding an available expression for which scalar quotient it will, however, be in general necessary to find previously the geometrical manner of motion of the body, or the law of the succession of the positions of that body or system in *space*. It may also be noticed here, that the comparison of the integrals (14) and (15) gives generally the relation:

$$S \,.\, \iota\gamma + h^2 = \sum . \, m \, S \int \iota\alpha\phi \, \mathrm{d}t. \tag{20}$$

III. When no accelerating forces are applied, or when such forces balance each other, we may treat the vector ϕ as vanishing, in the equations of the last section of this abstract; which thus become, for the *unaccelerated rotation of a solid body about a fixed point*, the following:

$$\sum . \, m\alpha V \,.\, \iota\alpha = \gamma; \tag{21}$$

$$\sum . \, m(V \,.\, \iota\alpha)^2 = -h^2; \tag{22}$$

$$\sum . \, m\alpha V \,.\, \alpha \, \mathrm{d}\iota = V \,.\, \iota\gamma \, \mathrm{d}t; \tag{23}$$

which result from (14) (15) (18), by supposing $\phi = 0$, or, more generally

$$\sum . \, mV \,.\, \alpha\phi = 0, \tag{24}$$

that is, by reducing the differential equation (12) of the second order, for the motion of the rigid system, to the form

$$\sum . \, mV \,.\, \alpha \frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2} = 0. \tag{25}$$

At the same time the general relation (20) reduces itself to the following:

$$S \,.\, \iota\gamma + h^2 = 0; \tag{26}$$

which may accordingly be obtained by a combination of the integrals (21) and (22); and the vector part of the quaternion $\iota\gamma$, of which the scalar part is thus $= -h^2$, may be expressed by means of the formula:

$$2V \,.\, \iota\gamma = V \sum . \, m(\iota\alpha)^2 = V \,.\, \iota \sum . \, m\alpha\iota\alpha; \tag{27}$$

which gives, by one of the transformations (17),

$$V \,.\, \iota\gamma = V \,.\, \iota \sum . \, m\alpha S \,.\, \alpha\iota; \tag{28}$$

so that we have, by (13) and (23),

$$\sum . \, m\alpha V \,.\, \alpha \, \mathrm{d}\iota = \sum . \, m \, \mathrm{d}\alpha \, S \,.\, \alpha\iota. \tag{29}$$

But also, by (21), because $S \,.\, \iota \, \mathrm{d}\alpha = 0$, we have

$$\sum . \, m\alpha V \,.\, \alpha \, \mathrm{d}\iota = \sum . \, m \, \mathrm{d}\alpha \, V \,.\, \alpha\iota + \sum . \, m\alpha\iota \, \mathrm{d}\alpha;$$

we ought, therefore, to find that

$$\sum . \, m(\mathrm{d}\alpha \,.\, \alpha\iota - \alpha\iota \,.\, \mathrm{d}\alpha) = 0,$$

or that

$$0 = V \sum . \, m(V \,.\, \iota\alpha \,.\, \mathrm{d}\alpha); \tag{30}$$

which accordingly is true, by (13), and may serve as a verification of the consistency of the foregoing calculations.

IV. We propose now briefly to point out a few of the *geometrical* consequences of the formulaæ in the foregoing section, and thereby to deduce, in a new way, some of the known properties of the rotation to which they relate; and especially to arrive anew at some of the theorems of Poinsot and Mac Cullagh. And first, it is evident on inspection that the equation (22) expresses that *the axis ι of instantaneous rotation is a semidiameter of a certain ellipsoid, fixed in the body, but moveable with it*; and having this property, that if the constant living force h^2 be divided by the square of the length of any such semidiameter ι, the quotient is the *moment of inertia of the body with respect to that semidiameter as an axis*: since the general rules of this calculus, when applied to the formula (22), give for this quotient the expression,

$$\sum . m(TV . \alpha U\iota)^2 = -h^2\iota^{-2} = h^2 T\iota^{-2}; \tag{31}$$

where $TV . \alpha U\iota$ denotes the length of the perpendicular let fall, on the axis ι, from the extremity of the vector α, that is, from the point or element of the body of which the mass is m. In the next place, the equation (26), which is of the first degree in ι, may be regarded as representing the *tangent plane* to the ellipsoid (22), at the extremity of the semidiameter ι; because this equation is satisfied by that semidiameter or vector ι, when we attribute to it the same value (in length and in direction) as before; and because if we change this vector ι to any infinitely near vector $\iota + \delta\iota$, consistent with the equation (22) of the ellipsoid, this near value of the vector will also be compatible with the equation (26) of the plane; for when the variation of the equation (22) is thus taken (by the rules of the present calculus), and is combined with the equation (21), it agrees with the equation (26) in giving

$$S . \gamma\,\delta\iota = 0. \tag{32}$$

But the plane (26) is *fixed in space*, on account of the constant vector γ and the constant scalar h, which were introduced by integration as above; consequently *the ellipsoid* (22) *rolls* (without gliding) *on the fixed plane* (26), *carrying with it the body in its motion*, and having its centre fixed at the fixed point of that body, or system, while the *semidiameter of contact ι* represents, in length and in direction, the *axis of a momentary rotation.* This is only a slightly varied form of a theorem discovered by Poinsot, which is one of the most beautiful of the results wherewith science has been enriched by that geometer: for the ellipsoid (22), which has here presented itself as a mode of *constructing the integral equation which expresses the law of living force of the system*, and which might for that reason be called the *ellipsoid of living force*, is easily seen to be concentric with, and similar to, the *central ellipsoid* of Poinsot, and to be similarly situated in the body. It may, however, be regarded as a somewhat remarkable circumstance, and one characteristic of the present method of calculation, that *it has not been necessary, in the foregoing process, to make any use of the three axes of inertia*, nor even to assume any knowledge of the *existence* of those three important axes; nor to make any other reference to any *axes of co-ordinates* whatsoever. The result of the calculation might be expressed by saying that "the ellipsoid of living force rolls on a plane parallel to the plane of areas;" and nothing farther, at this stage, might be supposed known respecting that ellipsoid (22), or respecting any other ellipsoid, than that it is a closed surface represented by an equation of the second degree. With respect to the *path of the axis of momentary rotation ι, within the body*, it is evident, from the equations (21), (22), that this path, or locus, is a *cone of the second degree*, which has for its equation the following:

$$\gamma^2 \sum . m(V . \iota\alpha)^2 = -h^2(\sum . m\alpha V . \iota\alpha)^2; \tag{33}$$

where the symbol γ^2, by one of the fundamental principles of the present calculus, is a certain negative scalar, namely, the negative of the square of the number which expresses the length of the vector γ, and which (in the present question) is constant by the law of the areas. Thus, according to another of Poinsot's modes of presenting to the mind a sensible image of the motion of the body, that motion of rotation may be conceived as the *rolling of a cone*, namely, of this cone (33), which is fixed in the body, but moveable therewith, on a certain other cone, which is the fixed locus in space of the instantaneous axis ι.

V. But we might also inquire, what is the *relative locus*, or what is the path *within* the body, of the vector γ, which has, by the law of areas, *a fixed direction*, as well as a *fixed length in space*: and thus we should be led to reproduce some of the theorems discovered by Mac Cullagh, in connexion with this celebrated problem of the rotation of a solid body. The equations (26) and (32) would give this other formula,

$$S \,.\, \iota\,\delta\gamma = 0; \tag{34}$$

and thus would shew that the vector γ is (in the body) a variable semidiameter of an ellipsoid *reciprocal* to that ellipsoid (22) of which the vector ι has been seen to be a semidiameter; and that these two vectors γ and ι are *corresponding semidiameters* of those two ellipsoids. The tangent plane to the new ellipsoid, at the extremity of the semidiameter γ (which extremity is fixed in space, but moveable within the body), is perpendicular to the axis ι of instantaneous rotation, and intercepts upon that axis a portion (measured from the centre) which has its length expressed by $h^2 T\iota^{-1}$, and which is, therefore, inversely proportional to the momentary and angular velocity (denoted here by $T\iota$), as it was found by Mac Cullagh to be. To find the *equation* of this reciprocal ellipsoid we have only to deduce, by the processes of this calculus, from the linear equation (21), an expression for the vector γ in terms of the vector ι, and then to substitute this expression in the equation (26). Making, for abridgment,

$$\left.\begin{aligned} n^2 &= -\textstyle\sum \,.\, m\alpha^2; \\ n'^2 &= -\textstyle\sum \,.\, mm'(V \,.\, \alpha\alpha')^2; \\ n''^2 &= +\textstyle\sum \,.\, mm'm''(S \,.\, \alpha\alpha'\alpha'')^2; \end{aligned}\right\} \tag{35}$$

so that n, n', n'', are real or scalar quantities, because the square of a vector is negative, and introducing a characteristic of operation σ, defined by the symbolic equation,

$$\sigma = \textstyle\sum \,.\, m\alpha S \,.\, \alpha, \quad \text{or} \quad \sigma\iota = \textstyle\sum \,.\, m\alpha S \,.\, \alpha\iota; \tag{36}$$

it is not difficult to show, first, that

$$(\sigma^2 + n^2\sigma + n'^2)\iota = -\textstyle\sum \,.\, mm'V \,.\, \alpha\alpha' \,.\, S \,.\, \alpha\alpha'\iota; \tag{37}$$

and then that the *symbol σ is a root of the symbolic and cubic equation*,

$$\sigma^3 + n^2\sigma^2 + n'^2\sigma + n''^2 = 0; \tag{38}$$

in the sense that *the operation denoted by the first member of this symbolic equation* (38) *reduces every vector ι, on which it is performed, to zero.* But the linear equation (21) may be thus written:

$$(\sigma + n^2)\iota = \gamma; \tag{39}$$

it gives, therefore, by (38),

$$(n^2 n'^2 - n''^2)\iota = (\sigma^2 + n'^2)\gamma; \tag{40}$$

that is, by (37) and (36),

$$(n''^2 - n^2 n'^2)\iota = n^2 \textstyle\sum . m\alpha S . \alpha\gamma + \sum . mm' V . \alpha\alpha' S . \alpha\alpha'\gamma. \tag{41}$$

Such being, then, the solution of this linear equation (21) or (39), the sought equation of Mac Cullagh's ellipsoid becomes, by (26),

$$(n^2 n'^2 - n''^2)h^2 = n^2 \textstyle\sum . m(S . \alpha\gamma)^2 + \sum . mm'(S . \alpha\alpha'\gamma)^2; \tag{42}$$

and we see that the following inequality must hold good:

$$n^2 n'^2 - n''^2 > 0. \tag{43}$$

If then a new and constant scalar g be determined by the condition,

$$(n^2 n'^2 - n''^2)h^2 + g^2\gamma^2 = 0, \tag{44}$$

(where γ^2 is still equal to the same constant and negative scalar as before), we may represent the *internal conical path*, or relative locus, of the vector γ in the body, by the equation:

$$0 = g^2\gamma^2 + n^2 \textstyle\sum . m(S . \alpha\gamma)^2 + \sum . mm'(S . \alpha\alpha'\gamma)^2. \tag{45}$$

We see then, by this analysis, that *the straight line γ which is drawn through the fixed centre of rotation, perpendicular to the plane of areas, describes within the body another cone of the second degree*: while the *extremity* of the same vector γ, which is a *fixed point in space, describes, by its relative motion, a spherical conic in the body*, namely, the curve of intersection of the cone (45) and the sphere (44): which agrees with Mac Cullagh's discoveries. Again the normal to the cone (45), which corresponds to the side γ, has the direction of the vector determined by the following expression:

$$\theta = \iota + h^2\gamma^{-1}; \tag{46}$$

and this new vector θ is always situated in the plane of areas, and is the side of contact of that plane with another cone of the second degree in the body, which is *reciprocal* to the cone (45), and was studied by both Poinsot and Mac Cullagh. But it would far exceed the limits of the present communication, if the author were to attempt here to call into review the labours of all the eminent men who, since the time of Euler, have treated, in their several ways, of the rotation of a solid body. He desires, however, before he concludes this sketch, to show how his own methods may be employed to assign the values of the three principal moments, and the positions of the three principal axes of inertia; although it has not been necessary for him, so far, on the plan which he has pursued, to make any use of those axes.

VI. Let us, then inquire under what conditions the body can continue to revolve, with a constant velocity, round a permanent axis of rotation. The condition of such a *double* permanence, of both the direction and the velocity of rotation, is completely expressed, on the present plan, by the one differential equation,

$$\frac{\mathrm{d}\iota}{\mathrm{d}t} = 0; \tag{47}$$

that is, in virtue of the formula (23), by

$$V \,.\, \iota\gamma = 0; \tag{48}$$

or, on account of (28) and (36), by this other equation,

$$(\sigma + s)\iota = 0, \tag{49}$$

where σ is the characteristic of operation lately employed, and s is a scalar coefficient, which must, if possible, be so determined as to allow the following symbolic expression for the sought permanent axis of rotation, namely,

$$\iota = (\sigma + s)^{-1}0, \tag{50}$$

to give a value different from zero,or to represent an actual vector ι, and not a null one. Now if we assumed any actual vector κ, such that

$$(\sigma + s)\iota = \kappa, \tag{51}$$

we should find, by the foregoing Section of this Abstract, and especially by the equations (37) and (38), a result of the form,

$$(s^3 - n^2 s^2 + n'^2 s + n''^2)\iota = \sigma'\kappa, \tag{52}$$

where σ' is a new characteristic of operation, such that

$$\sigma' = \sigma^2 - s\sigma + s^2 + n^2(\sigma - s) + n'^2, \tag{53}$$

and that, therefore,

$$\sigma'\kappa = s^2\kappa + s\textstyle\sum . m\alpha V \,.\, \alpha\kappa - \sum . mm' V \,.\, \alpha\alpha' S \,.\, \alpha\alpha'\kappa; \tag{54}$$

so that the solution (41) of the linear equation (39) is included in this more general result, which gives, for any arbitrary value of the number s the symbolic expression:

$$(\sigma + s)^{-1} = (s^3 - n^2 s^2 + n'^2 s + n''^2)^{-1}\sigma'. \tag{55}$$

Hence the condition for the non-evanescence of the expression (50), or the distinctive character of the permanent axes of rotation, is expressed by the cubic equation,

$$s^3 - n^2 s^2 + n'^2 s - n''^2 = 0. \tag{56}$$

The inequality (43) shows immediately that this equation (56) is satisfied by at least *one* real value of s, between the limits 0 and n^2; and an attentive examination of the composition (35) of the coefficients of the same cubic equation in s, would prove that this cubic has in general *three* real and unequal roots, between the same two limits; which roots we may denote by s_1, s_2, s_3. Assuming next any *arbitrary* vector κ, and deriving from it two other vectors, κ' and κ'', by the formulæ

$$\textstyle\sum . m\alpha V . \alpha\kappa = \kappa'; \quad -\sum . mm'V . \alpha\alpha' S . \alpha\alpha'\kappa = \kappa''; \tag{57}$$

making also

$$\left.\begin{aligned} \iota_1 &= s_1^2\kappa + s_1\kappa' + \kappa'', \\ \iota_2 &= s_2^2\kappa + s_2\kappa' + \kappa'', \\ \iota_3 &= s_3^2\kappa + s_3\kappa' + \kappa''; \end{aligned}\right\} \tag{58}$$

we shall thus have, in general, a system of three rectangular vectors, ι_1, ι_2, ι_3, in the directions of the *three principal axes*. For first they will be, by (54), the three results of the form $\sigma'\kappa$, obtained by changing s, successively and separately, to the three roots of the ordinary cubic (56); but by the manner of dependence (53) or the characteristic σ' on σ and s, and by the symbolic equation of cubic form (38) in σ, we have, if s be any one of those three roots of (56), the relation

$$(\sigma + s)\sigma'\kappa = 0; \tag{59}$$

consequently the three vectors (58) are such that

$$0 = (\sigma + s_1)\iota_1 = (\sigma + s_2)\iota_2 = (\sigma + s_3)\iota_3. \tag{60}$$

Each of the vectors ι_1, ι_2, ι_3, is therefore, by (49), adapted to become a permanent axis of rotation of the body; while the foregoing analysis shows that in general no other vector ι, which has not the direction of one of those three vectors (58), or an exactly opposite direction, is fitted to become an axis of such permanent rotation. And to prove that these three axes are in general at right angles to each other, or that they satisfy in general the three following equations of perpendicularity,

$$0 = S . \iota_1\iota_2 = S . \iota_2\iota_3 = S . \iota_3\iota_1, \tag{61}$$

we may observe that, for any two vectors ι, κ, the form (36) of the characteristic σ gives,

$$S . \kappa\sigma\iota = \textstyle\sum . mS . \kappa\alpha S . \alpha\iota = S . \iota\sigma\kappa, \tag{62}$$

and therefore, for any scalar s,

$$S . \kappa(\sigma + s)\iota = S . \iota(\sigma + s)\kappa; \tag{63}$$

consequently the two first of the equations (60) give (by changing ι, κ, s to ι_2, ι_1, s_1),

$$(s_1 - s_2)S . \iota_1\iota_2 = 0; \tag{64}$$

and therefore they conduct to the first equation of perpendicularity (61), or serve to show that the two axes, ι_1 and ι_2, are mutually rectangular, at least in the general case, when the two corresponding roots, s_1 and s_2, of the equation (56), are unequal. The equations (48) and (32), namely, $V \,.\, \iota\gamma = 0$, $S \,.\, \gamma\,\delta\iota = 0$, show also that these three rectangular axes of inertia are in the directions of the *axes of the ellipsoid* (22), which has presented itself as a sort of construction of the law of living force of the system; and a *common property* of these three rectangular directions, which in general belongs *exclusively* to them, and to their respectively opposite directions, may be expressed by the rules of this calculus under the very simple form,

$$0 = V \sum . \, m(\iota\alpha)^2; \tag{65}$$

or under the following, which is equivalent thereto,

$$\sum . \, m(\iota\alpha)^2 = \sum . \, m(\alpha\iota)^2. \tag{66}$$

With respect to the geometrical and physical significations of the three values of the positive scalar s, the equation (49) gives

$$s\iota^2 + S \,.\, \iota\sigma\iota = 0; \tag{67}$$

and consequently by (36), and by the general rules of this calculus,

$$s = \sum . \, m(S \,.\, \alpha\, U\iota)^2 = \sum . \, mx^2, \tag{68}$$

if x denote the perpendicular distance of the mass m from the plane drawn through the fixed point of the body, in a direction perpendicular to the axis ι. We may therefore write the follow expressions for the three roots of the cubic (56):

$$s_1 = \sum . \, mx^2; \quad s_2 = \sum . \, my^2; \quad s_3 = \sum . \, mz^2; \tag{69}$$

if xyz denote (as usual) three rectangular coordinates, of which the axes here coincide respectively with the directions of ι_1, ι_2, ι_3; and we see that the *three principal moments* of inertia, or those relative to these three axes, are the three sums,

$$s_2 + s_3, \quad s_3 + s_1, \quad s_1 + s_2, \tag{70}$$

of pairs of roots of the cubic equation which has been employed in the present method. At the same time, the conditions above assigned for the directions of those three axes take easily the well-known forms,

$$0 = \sum . \, mxy = \sum . \, myz = \sum . \, mzx, \tag{71}$$

if (for the sake of comparison with known results) we change the vectors $\alpha, \alpha', \ldots$ of the masses $m, m', \ldots$ to the expressions

$$\alpha = ix + jy + kz, \quad \alpha' = ix' + jy' + kz', \ldots \tag{72}$$

where xyz are the rectangular co-ordinates of m, and ijk are the three original and fundamental symbols of the present Calculus, denoting generally three rectangular vector-units, and subject to the laws of symbolical combination which were communicated to the Academy by the author in 1843, and are included in the formula (4) of the present Abstract. And then, by (35), the coefficients of the cubic equation (56) will take the following forms, which easily admit of being interpreted, or of being translated into geometrical enunciations:

$$\left.\begin{aligned} n^2 &= \textstyle\sum . m(x^2+y^2+z^2); \\ n'^2 &= \textstyle\sum . mm'\{(yz'-zy')^2+(zx'-xz')^2+(xy'-yx')^2\}; \\ n''^2 &= \textstyle\sum . mm'm''\{(yz'-zy')x''+(zx'-xz')y''+(xy'-yx')z''\}^2. \end{aligned}\right\} \quad (73)$$

In fact, the first of these three expressions is evidently the sum of the three quantities (69); and it is not difficult to prove that, under the conditions (71), the second expression (73) is equal to the sum of the three binary products of those three quantities; and that the third expression (73) is equal to their continued or ternary product: in such manner as to give

$$\left.\begin{aligned} s_1+s_2+s_3 &= n^2; \\ s_1s_2+s_2s_3+s_3s_1 &= n'^2; \\ s_1s_2s_3 &= n''^2. \end{aligned}\right\} \quad (74)$$

Perhaps, however, it may not have been noticed before, that expressions possessing so *internal* a character as do these three expressions (73), and admitting of such simple *interpretations* as they do, without any *previous* reference to the axes of inertia, or indeed to *any axes* (since all is seen to depend on the *masses and mutual distances* of the several points or elements of the system), are the coefficients of a cubic equation which has the well-known sums, $\sum .mx^2$, $\sum .my^2$, $\sum .mz^2$, referred to the three principal planes, for its three roots. In the method of the present communication, those expressions (73), or rather the more concise but equivalent expressions (35), have been seen to offer themselves as coefficients of a *symbolic equation of the third degree* (38), which is satisfied by a certain *characteristic of operation* σ, connected with the solution of a certain other symbolic but *linear* equation: and the Author may be permitted to mention that this is only a particular (though an important) application of a general method, which he has for a considerable time past possessed, for the solution of those linear equations to which the Calculus of Quaternions conducts. To those who have perused the foregoing sections of this Abstract, and who have also read with attention the Abstract of his communication of July, 1846, published in the Proceedings of that date, he conceives that it will be evident that *for any fixed point A of any solid body* (or rigid system), *there can be found* (indeed in more ways than one) *a pair of other points B and C, which are likewise fixed in the body, and are such that the square-root of the moment of inertia round any axis AD is geometrically constructed or represented by the line BD, if the points A and D be at equal distances from C.*

VII. Finally, he desires to mention here one other theorem respecting rotation, which is indeed more of a geometrical than of a physical character, and to which his own methods have led him. By employing certain general principles, respecting powers and roots, and respecting differentials and integrals of Quaternions, he finds that for any system or set of diverging vectors, $\alpha, \beta, \gamma, \dots \kappa, \lambda$, the continued product of the square roots of their successive quotients may be expressed under the following form:

$$\left(\frac{\alpha}{\beta}\right)^{\frac{1}{2}} \left(\frac{\beta}{\gamma}\right)^{\frac{1}{2}} \cdots \left(\frac{\kappa}{\lambda}\right)^{\frac{1}{2}} \left(\frac{\lambda}{\alpha}\right)^{\frac{1}{2}} = (\cos \pm U\alpha \sin)\frac{s}{2}; \tag{75}$$

where s is a scalar which represents the *spherical excess* of the pyramidal angle formed by the diverging vectors; or the *spherical opening* of that pyramid; or the *area* of the spherical polygon, of which the corners are the points where the vectors $\alpha, \beta, \gamma, \dots, \kappa, \lambda$, meet the spheric surface described about their common origin with a radius equal to unity. And by combining this result with the general method stated to the Academy by the Author* in November, 1844, for connecting quaternions with rotations, it is easy to conclude that if a solid body be made to revolve in succession round any number of different axes, all passing through one fixed point so as first to bring a line α into coincidence with a line β, by a rotation round an axis perpendicular to both; secondly, to bring the line β into coincidence with a line γ, by turning round an axis to which both β and γ are perpendicular; and so on, till, after bringing the line κ to the position λ, the line λ is brought to the position α with which we began; then *the body will be brought, by this succession of rotations, into the same final position as if it had revolved round the first or last position of the line α, as an axis, through an angle of finite rotation, which has the same numerical measure as the spherical opening of the pyramid ($\alpha, \beta, \gamma, \dots \kappa, \lambda$) whose edges are the successive positions of that line.*

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* The same connexion between the Author's principles, and geometrical and algebraical questions, respecting the rotation of a solid body, or respecting the closely connected subject of the transformation of rectangular coordinates, was independently perceived by Mr. Cayley; who inserted a communication on the subject in the Philosophical Magazine for February, 1845, under the title, "Results respecting Quaternions." It is impossible for the Author, in the present sketch, to do more than refer here to Mr. Cayley's important researches respecting the Dynamics of Rotation, published in the Cambridge and Dublin Mathematical Journal. An account of the speculations and results of the late Professor Mac Cullagh on this subject may be found in part viii. of the Proceedings of the Royal Irish Academy; and a summary of the views and discoveries of Poinsot has been given by that able author in his very interesting tract, entitled, *Théorie Nouvelle de la Rotation des Corps*, Paris, 1834.