ON RÖBER'S CONSTRUCTION OF THE HEPTAGON

By

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On Röber's *Construction of the Heptagon. By* Sir William Rowan Hamilton, *LL.D., M.R.I.A., F.R.A.S., &c., Andrews Professor of Astronomy in the University of Dublin, and Royal Astronomer of Ireland**.

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1. In a recent Number of the Philosophical Magazine, observations were made on some approximate constructions of the regular heptagon, which have recalled my attention to a very remarkable construction of that kind, invented by a deceased professor of architecture at Dresden, Friedrich Gottlob Röber†, who came to conceive, however, that it had been known to the Egyptians, and employed by them in the building of the temple at Edfu. Röber, indeed, was of opinion that the connected triangle, in which each angle of the base is triple of the angle at the vertex, bears very important relations to the plan of the human skeleton, and to other parts of nature. But without pretending to follow him in such speculations, attractive as they may be to many readers, I may be permitted to examine here the accuracy of the proposed geometrical construction, of such an isosceles triangle, or of the heptagon which depends upon it. The closeness of the approximation, although short of mathematical rigour, will be found very surprising.

2. Röber's diagram is not very complex, and may even be considered to be elegant; but the essential parts of the construction are sufficiently expressed by the following formulæ: in which p denotes a side of a regular pentagon; r, r' the radii of its inscribed and circumscribed circles; r'' the radius of a third circle, concentric with but exterior to both; p' a segment of the side p; and q, s, t, u, v five other derived lines. The result is, that in the right-angled triangle of which the inner diameter $2r$ is the hypotenuse, and u, v supplementary chords, the former chord (u) is *very nearly* equal to a side of a regular heptagon, inscribed in the interior circle; while the latter chord (v) makes with the diameter ($2r$) an angle ϕ, which is

* Communicated by the Author.

† The construction appears to have been first given in pages 15, 16 of a quarto work by his son, Friedrich Röber, published at Dresden in 1854, and entitled *Beiträge zur Erforschung der geometrischen Grundformen in den alten Tempeln Aegyptens, und deren Beziehung zur alten Naturerkenntniss.* It is repeated in page 20 of a posthumous work, or collection of papers, edited by the younger Röber, and published at Leipzig in 1861, entitled *Elementar-Beiträge zur Bestimmung des Naturgesetzes der Gestaltung und des Widerstandes, und Anwendung dieser Beiträge auf Natur und alte Kunstgestaltung*, von Friedrich Gottlob Röber, ehemaligen Königlich-Sächsischen Professor der Baukunst und Land-Baumeister. Both works, and a third upon the pyramids, to which I cannot at present refer, are replete with the most curious speculations, into which however I have above declined to enter.

very nearly equal to the vertical angle of an isosceles triangle, whereof each angle at the base is triple of the angle at the vertex. In symbols, if we write

$$u = 2r \sin\phi, \quad v = 2r \cos\phi,$$

then ϕ is found to be very nearly $= \dfrac{\pi}{7}$. It will be seen that the equations can all be easily constructed by right lines and circles alone, having in fact been formed as the expression of such a construction; and that the numerical ratios of the lines, including the numerical values of the sine and cosine of ϕ, can all be arithmetically computed*, with a few extractions of square roots.

$$(\mathrm{A}) \quad \begin{cases} (r+r')^2 = 5r^2 & \dfrac{r'}{r} = 1.2360680 \\ p^2 = 4(r'^2 - r^2) & \dfrac{p}{r} = 1.4530851 \\ \dfrac{p'}{p} = \dfrac{r + \frac{1}{2}r'}{r+r'} & \dfrac{p'}{r} = 1.0514622 \\ q^2 = p^2 - p'^2 & \dfrac{q}{r} = 1.0029374 \\ s^2 + ps = (p - q + r)^2 & \dfrac{s}{r} = 0.8954292 \\ r''^2 = r^2 + s^2 & \dfrac{r''}{r} = 1.3423090 \\ t^2 = \left(\dfrac{r'r''}{r}\right)^2 - (r'' - r)^2 & \dfrac{t}{r} = 1.6234901 \\ u^2 = 2r(2r - t) & \dfrac{u}{r} = 0.8677672 \\ v^2 = 2rt & \dfrac{v}{r} = 1.8019379 \\ u = 2r\sin\phi & \sin\phi = 0.4338836 \\ v = 2r\cos\phi & \cos\phi = 0.9009689 \end{cases}$$

3. On the other hand, the true septisection of the circle may be made to depend on the solution of the cubic equation,

$$8x^3 + 4x^2 - 4x - 1 = 0,$$

of which the roots are $\cos\dfrac{2\pi}{7}$, $\cos\dfrac{4\pi}{7}$, $\cos\dfrac{6\pi}{7}$. Calculating then, by known methods†, to eight decimals, the positive root of this equation, and thence deducing to seven decimals, by square

* The computations have all been carried out to several decimal places beyond what are here set down. Results of analogous calculations have been given by Röber, and are found in page 16 of the first-cited publication of his son, with the assumption $p = \sqrt{3}$, and with one place fewer of decimals.

† Among these the best by far appears to be Horner's method,—for practically arranging

roots, the sine and cosine of $\frac{\pi}{7}$, we find, without tables, the values

$$\cos\frac{2\pi}{7} = \qquad x = 0.62348980;$$
$$\sin\frac{\pi}{7} = \sqrt{\frac{1-x}{2}} = 0.4338837;$$
$$\cos\frac{\pi}{7} = \sqrt{\frac{1+x}{2}} = 0.9009689;$$

and these last agree so nearly with the values (A) of $\sin\phi$ and $\cos\phi$, that at this stage a doubt may be felt, *in which direction does the construction err*. In fact, Röber appears to have believed that the construction described above was geometrically rigorous, and had been known and prized as such from a very remote antiquity, although preserved as a secret doctrine, entrusted only to the initiated, and recorded only in stone.

4. The following is an easier way, for a reader who may not like so much arithmetic, to satisfy himself of the extreme closeness of the approximation, by formulæ adapted to logarithms, but rigorously derived from the construction. It being evident that

$$r' = r\sec\frac{\pi}{5}, \quad \text{and } p = 2r\tan\frac{\pi}{5},$$

let $\phi_1 \ldots \phi_6$ be six auxiliary angles, such that

$$r' = 2r\tan\phi_1, \quad p' = p\sin 2\phi_2, \quad p - q = r\tan^2\phi_3,$$

$$p - q + r = \tfrac{1}{2}p\tan 2\phi_4, \quad s = r\tan 2\phi_5, \quad r(r'' - r) = r'r''\sin\phi_6;$$

we shall then have the following system of equations, to which are annexed the angular values, deduced by interpolation from Taylor's seven-figure logarithms, only eleven openings of which

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the figures in the use of which method, a very compact and convenient form or scheme was obligingly communicated to me by Professor De Morgan, some time ago. We arrived independently at the following value, to 22 decimals, of the positive root of the cubic mentioned above:

$$\cos\frac{2\pi}{7} = 0\cdot 62348\ 98018\ 58733\ 53052\ 50.$$

I had however found, by trials, before using Horner's method, the following approximate value:

$$\cos\frac{2\pi}{7} = 0\cdot 62348\ 98018\ 587;$$

which was more than sufficiently exact for comparison with Röber's construction.

are required, if the logarithms of two and four be remembered, as they cannot fail to be by every calculator.

$$
\text{(B)} \quad \begin{cases}
\cot\phi_1 = 2\cos\dfrac{\pi}{5} & \phi_1 = 31^\circ\ 43'\ \ 2''\cdot 91 \\
\sin 2\phi_2 = \cos^2\phi_1 & \phi_2 = 23\ 10\ 35\cdot 52 \\
\tan^2\phi_3 = 4\sin^2\phi_2\ \tan\dfrac{\pi}{5} & \phi_3 = 33\ 51\ 31\cdot 90 \\
\cot 2\phi_4 = \cos^2\phi_3\ \tan\dfrac{\pi}{5} & \phi_4 = 31\ 41\ 39\cdot 37 \\
\tan 2\phi_5 = 2\sin^2\phi_4\ \sec 2\phi_4\ \tan\dfrac{\pi}{5} & \phi_5 = 20\ 55\ 15\cdot 93 \\
\sin\phi_6 = \sin^2\phi_5\ \cot\phi_1 & \phi_6 = 11\ 54\ 22\cdot 60 \\
\cos^2\phi = \cos\phi_6\ \sec 2\phi_5\ \tan\phi_1 & \phi = 25\ 42\ 51\cdot 4\,.
\end{cases}
$$

It is useless to attempt to estimate hundredths of seconds in this last value, because the difference for a second, in the last logarithmic cosine, amounts only to ten units in the seventh place of decimals, or to one in the sixth place. But if we thus confine ourselves to tenths of seconds, a simple division gives immediately that final value, under the form

$$\frac{\pi}{7} = \frac{180^\circ}{7} = 25^\circ\ 42'\ 51''\cdot 4;$$

it appears therefore to be difficult, if it be possible, to decide by Taylor's tables, whether the equations (B), deduced from Röber's construction, give a value of the angle ϕ, which is *greater* or *less* than the *seventh part of two right angles.* (It may be noted that $2\tan\phi_1 = 2$; but that to take out ϕ_1 by this equation would require another opening of the tables.)

5. To fix then decisively the *direction* of the *error* of the approximation, and to form with any exactness an estimate of its *amount*, or even to prove quite satisfactorily by *calculation* that any such error *exists*, it becomes necessary to fall back on arithmetic; and to carry at least the first extractions to several more places of decimals,—although fewer than those which have been used in the resumed computation might have sufficed, except for the extreme accuracy aimed at in the resulting values. For this purpose, it has been thought convenient to introduce eight auxiliary numbers $a \ldots h$, which can all be calculated by square roots, and are defined with reference to the recent equations (B), as follows:

$$a = 1 + 2\tan\phi_1; \quad b = 4\cos 2\phi_2 \cot\frac{\pi}{5}; \quad c = 2\cos 2\phi_2 - \cot\frac{\pi}{5};$$

$$d = \sec 2\phi_4; \quad e = \sec 2\phi_5; \quad f = 2\cos^2\phi; \quad g = 2\cos\phi; \quad h = 2\cos\frac{\phi}{2};$$

or thus with reference to the earlier equations (A):

$$a = \frac{r + r'}{r}; \quad b = \frac{8qr}{p^2}; \quad c = \frac{2(q - r)}{p}; \quad d = \frac{2s + p}{p}; \quad e = \frac{r''}{r};$$

$$f = \frac{t}{r}; \quad f = \frac{v}{r}; \quad h^2 = \frac{2r+v}{r};$$

and respecting which it is to be observed that c, like the rest, is positive, because it may be put under the form

$$c = \sqrt{\frac{14 - 2\sqrt{5}}{5}} - \sqrt{\frac{5 + 2\sqrt{5}}{5}},$$

and $14 - 2\sqrt{5} > 5 + 2\sqrt{5}$, because $9 > 4\sqrt{5}$, or $9^2 > 4^2\ 5$. With these definitions, then, of the numbers $a \ldots h$, and with the help of the following among other identities,

$$\begin{aligned} \cos\frac{7\phi}{2}\ \sec\frac{\phi}{2} &= 2\cos 3\phi - 2\cos 2\phi + 2\cos\phi - 1 \\ &= 2(2\cos\phi - 1)\cos 2\phi - 1, \end{aligned}$$

I form *without tables* a system of values below, the early numbers of which have been computed to several decimals more than are set down.

$$\text{(C)} \quad \begin{cases} a^2 = 5 & a = 2 \cdot 23606\ 79774\ 99789\ 6964 \\ b^2 = 8 + \dfrac{72a}{25} & b = 3 \cdot 79998\ 36545\ 96345\ 0138 \\ c^2 = \dfrac{19}{5} - b & c = 0 \cdot 00404\ 29449\ 23565\ 7641 \\ d^2 = 1 + (2-c)^2 & d = 2 \cdot 23245\ 25898\ 01044\ 7849 \\ e^2 = 1 + (5-2a)(d-1)^2 & e = 1 \cdot 34230\ 90137\ 74792\ 5831 \\ f^2 = (5-2a)e^2 + 2e - 1 & f = 1 \cdot 62349\ 00759\ 24105\ 2470 \\ g^2 = 2f & g = 1 \cdot 80193\ 78878\ 99638\ 5912 \\ h^2 = 2 + g & h = 1 \cdot 94985\ 58633\ 65197\ 2049 \\ \sin\dfrac{\pi - 7\phi}{2} = h\left((f-1)(g-1) - \dfrac{1}{2}\right) & \quad = +0 \cdot 00000\ 06134\ 49929\ 1683. \end{cases}$$

Admitting then the known value,

$$\pi = 3 \cdot 14159\ 26535\ 89793\ldots,$$

or the deduced expression,

$$1'' = \frac{\pi}{648000} = 0 \cdot 00000\ 48481\ 36811\ 095\ldots,$$

I infer as follows:

$$\text{(D)} \quad \begin{cases} \dfrac{\pi - 7\phi}{2} = +0'' \cdot 12653\ 31307\ 822, \\ \dfrac{\pi}{7} - \phi = +0'' \cdot 03615\ 23230\ 806, \\ \phi = 25^\circ\ 42'\ 51'' \cdot 39241\ 91054\ 91, \end{cases}$$

and think that these twelve decimals of a second, in the value of the angle ϕ, may all be relied on, from the care which has been taken in the calculations.

6. The following is a quite different way, as regards the few last steps, of deducing the same ultimate numerical results. Admitting (comp. Art. 3) the value*,

$$2\cos\frac{2\pi}{7} = z = 1 \cdot 24697\ 96037\ 17467\ 06105,$$

as the positive root, computed by Horner's method, of the cubic equation

$$z^3 + z^2 - 2z - 1 = 0,$$

and employing the lately calculated value f of $1 + \cos 2\phi$, I find by square roots the following sines and cosines, with the same resulting error of the angle ϕ as before:

$$\text{(E)} \quad \begin{cases} \sin\dfrac{\pi}{7} = \dfrac{1}{2}\sqrt{2-z} = 0 \cdot 43388\ 37391\ 17558\ 1205; \\ \cos\dfrac{\pi}{7} = \dfrac{1}{2}\sqrt{2+z} = 0 \cdot 90096\ 88679\ 02419\ 1262; \\ \sin\phi = \dfrac{1}{2}\sqrt{4-2f} = 0 \cdot 43388\ 35812\ 03469\ 1138; \\ \cos\phi = \dfrac{1}{2}\sqrt{2f} = \dfrac{1}{2}g = 0 \cdot 90096\ 89439\ 49819\ 2956; \\ \sin\left(\dfrac{\pi}{7} - \phi\right) = +0 \cdot 00000\ 01752\ 71408\ 3339; \\ \dfrac{\pi}{7} - \phi = +0'' \cdot 03615\ 23230\ 806. \end{cases}$$

7. If we continue the construction, as Röber did, so as to form an isosceles triangle, say ABC, with ϕ for its vertical angle, and if we content ourselves with thousandths of seconds, the angles of this triangle will be as follows:

$$\text{(F)} \quad \begin{cases} \text{A} = \phi = 25^\circ\ 42'\ 51'' \cdot 392; \\ \text{B} = \dfrac{\pi - \phi}{2} = 77^\circ\ 8'\ 34'' \cdot 304; \\ \text{C} = \text{B} = 77^\circ\ 8'\ 34'' \cdot 304; \end{cases}$$

and we see that each base-angle exceeds the triple of the vertical by only about an eighth part of a second, namely by that small angle which occurs first in the system (D), and of which the sine is the last number in the preceding system (C). And if we compare a base-angle of the triangle thus constructed, with the base-angle $\frac{3\pi}{7} = 77^\circ\ 8'\ 34'' \cdot 2857\ldots$ of the true triangle, in which each angle of the base is triple of the angle at the vertex, we find an error in excess equal nearly to $0'' \cdot 018$, or, more exactly,

$$\text{B} - \frac{3\pi}{7} = +0'' \cdot 01807\ 61615\ 403,$$

* Compare a preceding note.

which amounts to *less* than a *fifth-fifth part* of a *second*, but of which I conceive that all the thirteen decimals here assigned are correct. And I suppose that no artist would undertake to construct a triangle which should more perfectly, or so perfectly, fulfil the conditions proposed. The *problem*, therefore, of constructing *such* a *triangle*, and with it the *regular heptagon*, by *right lines and circles only*, has been *practically solved* by that process which Röber believed to have been *known* to the ancient Egyptians, and to have been *employed* by them in the architecture of some of their temples—some *hints*, as he judged, being intentionally preserved in the details of the workmanship, for the purpose of being *recognized*, by the initiated of the time, or by men of a later age.

8. Another way of rendering conceivable the extreme smallness of the practical error of that process, is to imagine a series of seven successive chords inscribed in a circle, according to the construction in question, and to inquire *how near* to the initial point the final point would be. The answer is, that the *last* point would fall *behind* the *first*, but only by about *half a second* (more exactly by $0''\cdot 506$). If then we suppose, for illustration, that these chords are *sevven successive tunnels*, drawn *eastward* from station to station of the *equator of the earth*, the last tunnel would emerge to the *west* of the first station, but only by about *fifty feet*.

9. My own studies have not been such as to entitle me to express an opinion whether the architectural and geometrical drawings of Röber in connexion with the plan of the temple at Edfu, and his comparisons of the numbers deduced from the *details* of his construction with French measurements previously made, are sufficient to bear out his conclusion, that the process had been anciently used: but I wish that some qualified person would take up the inquiry, which appears to me one of great interest, especially as I see no antecedent improbability in the supposition that the construction in question may have been invented in a very distant age. The geometry which it employs is in no degree more difficult than that of the Fourth Book of Euclid*; and although I have no conjecture to offer as to what may have *suggested* the particular process employed, yet it seems to me quite as likely to have been discovered thousands of years ago, perhaps after centuries of tentation, as to have been first found in our own time, which does not generally attact so much importance to the heptagon as a former age may have done, and which perhaps enjoys no special facilities in the search after such a *construction*, although it supplies means of proving, as above, that the proposed solution of the problem is *not mathematically perfect*.

10. If Röber's professional skill as an architect, combined with the circumstance stated of his having previously invented the construction for himself did really lead him to a correct

* The segment p' of the side p of the pentagon, and the fourth proportional $\dfrac{r'r''}{r}$ to the three radii, which enter into the equations (A), and of which the latter is the greater segment of the third diameter, $2r''$, if this last be cut in extreme and mean ratio, may at first appear to depend on the Sixth Book of Euclid, but will be found to be easily constructible without going beyond the Fourth Book.

interpretation* of the plan of the temple at Edfu, which he believed to embody a tradition much older than itself, we are thus admitted to a very curious glimpse, or even a partial view, of the nature and extent, but at the same time the imperfection, of that knowledge of geometry which was possessed, but kept secret, by the ancient priests of Egypt. I say the *imperfection*, on the supposition that the above described construction of the *heptagon*, if known to them at all, was thought by them to be *equal in rigour*, as the elder Röber appears to have thought it to be, to that construction of the *pentagon* which Euclid *may* have learned from them, *rejecting* perhaps, at the same time, the *other* construction, as being not based on demonstration, and not by him demonstrable, although Euclid may not have *known* it to be, in its result, imperfect. The interest of the speculation stretches indeed back to a still earlier age, and may be connected in imagination with what we read of the "wisdom of the Egyptians." But I trust that I shall be found to have abstained, as I was bound to do, from any expression which could imply an acquaintance of my own with the archæology of Egypt, and that I may at least be pardoned, if not thanked, for having thus submitted, to those who may be disposed to study the subject, a purely mathematical† discussion, although connected with a question of other than mathematical interest.

W. R. H.

Observatory of Trinity College, Dublin,
December 22, 1863.

* It ought in fairness to be stated that Röber's *interpretation* of Egyptian antiquities included a vast deal more than what is here described, and that he probably considered the *geometrical part* of it to be the *least interesting*, although still, in his view, an essential and *primary element.*

† *Note added during printing.*—Some friends of the writer may be glad to know that these long arithmetical calculations have been to him rather a relaxation than a distraction from his more habitual studies, and that there are already in type 672 octavo pages of the 'Elements of Quaternions,' a work which (as he hopes) is rapidly approaching to the stage at which it may be announced for publication.