

**ON CERTAIN DISCONTINUOUS INTEGRALS
CONNECTED WITH THE DEVELOPMENT OF
THE RADICAL WHICH REPRESENTS THE
RECIPROCAL OF THE DISTANCE BETWEEN
TWO POINTS**

By

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(Philosophical Magazine, 20 (1842), pp. 288–294.)

Edited by David R. Wilkins

2000

On certain discontinuous Integrals, connected with the Development of the Radical which represents the Reciprocal of the Distance between two Points. By WILLIAM ROWAN HAMILTON, LL.D., P.R.I.A., Member of several Scientific Societies at Home and Abroad, Andrews' Professor of Astronomy in the University of Dublin, and Royal Astronomer of Ireland.*

[*The London, Edinburgh and Dublin Philosophical Magazine and Journal of Science*, 3rd series, vol. xx (1842), pp. 288–294.]

1. It is well known that the radical

$$(1 - 2xp + x^2)^{-\frac{1}{2}}, \quad (1.)$$

in which x and 1 may represent the radii vectores of two points, while p represents the cosine of the angle between those radii, and the radical represents therefore the reciprocal of the distance of the one point from the other, may be developed in a series of the form

$$P_0 + P_1x + P_2x^2 + \dots + P_nx^n + \dots; \quad (2.)$$

the coefficients P_n being functions of p , and possessing many known properties, among which we shall here employ the following only,

$$P_n = [0]^{-n} \left(\frac{d}{dp} \right)^n \left(\frac{p^2 - 1}{2} \right)^n; \quad (3.)$$

the known notation of factorials being here used, according to which

$$[0]^{-n} = \frac{1}{1} \cdot \frac{1}{2} \cdot \frac{1}{3} \cdots \frac{1}{n}. \quad (4.)$$

It is proposed to express the sum of the first n terms of the development (2.), which may be thus denoted,

$$\Sigma_{(n)} \binom{n-1}{0} P_n x^n = P_0 + P_1x + P_2x^2 + \dots + P_{n-1}x^{n-1}. \quad (5.)$$

2. In general, by Taylor's theorem,

$$f(p + q) = \Sigma_{(n)} \binom{\infty}{0} [0]^{-n} q^n \left(\frac{d}{dp} \right)^n f(p); \quad (6.)$$

* Communicated by the Author.

hence, by the property (3.), P_n is the coefficient of q^n in the development of

$$\left(\frac{(p+q)^2-1}{2}\right)^n; \quad (7.)$$

it is therefore also the coefficient of q^0 in the development of

$$\left(\frac{p^2-1}{2q} + p + \frac{q}{2}\right)^n \quad (8.)$$

If then we make, for abridgment,

$$\vartheta = p + \frac{p^2}{2} \cos \theta + \sqrt{-1} \left(1 - \frac{p^2}{2}\right) \sin \theta, \quad (9.)$$

we shall have the following expression, which perhaps is new, for P_n :

$$P_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \vartheta^n d\theta; \quad (10.)$$

and hence, immediately, the required sum (5.) may be expressed as follows:

$$\Sigma_{(n)} \binom{n-1}{0} P_n x^n = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \frac{1 - \vartheta^n x^n}{1 - \vartheta x}; \quad (11.)$$

in which it is to be observed that x may be any quantity, real or imaginary.

3. We have therefore, rigorously, for the sum of the n first terms of the series

$$P_0 + P_1 + P_2 + \dots, \quad (12.)$$

the expression

$$\Sigma_{(n)} \binom{n-1}{0} P_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \frac{1 - \vartheta^n}{1 - \vartheta}; \quad (13.)$$

of which we propose to consider now the part independent of n , namely,

$$F(p) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta}{1 - \vartheta}; \quad (14.)$$

and to examine the form of this function F of p , at least between the limits $p = -1$, $p = 1$.

4. A little attention shows that the denominator $1 - \vartheta$ may be decomposed into factors, as follows:

$$1 - \vartheta = -\frac{1}{2}(\alpha + e^{\theta\sqrt{-1}})(1 - \beta e^{-\theta\sqrt{-1}}); \quad (15.)$$

in which,

$$\alpha = 2s(1 - s), \quad \beta = 2s(1 + s), \quad (16.)$$

and

$$p = 1 - 2s^2; \quad (17.)$$

so that s may be supposed not to exceed the limits 0 and 1, since p is supposed not to exceed the limits -1 and 1. Hence

$$\frac{1}{1 - \vartheta} = \frac{-2(\alpha + e^{-\theta\sqrt{-1}})(1 - \beta e^{\theta\sqrt{-1}})}{(1 + 2\alpha \cos \theta + \alpha^2)(1 - 2\beta \cos \theta + \beta^2)}; \quad (18.)$$

of which the real part may be put under the form

$$\frac{\lambda}{1 + 2\alpha \cos \theta + \alpha^2} + \frac{\mu}{1 - 2\beta \cos \theta + \beta^2}, \quad (19.)$$

if λ and μ be so chosen as to satisfy the conditions

$$\lambda(1 + \beta^2) + \mu(1 + \alpha^2) = 2(\beta - \alpha), \quad (20.)$$

$$\lambda\beta - \mu\alpha = 1 - \alpha\beta, \quad (21.)$$

which give

$$\lambda = \frac{1 - \alpha^2}{\alpha + \beta}, \quad \mu = \frac{\beta^2 - 1}{\alpha + \beta}. \quad (22.)$$

The imaginary part of the expression (18.) changes sign with θ , and disappears in the integral (14.); that integral therefore reduces itself to the sum of the two following:

$$F(p) = \frac{1}{4s\pi} \int_0^\pi \frac{(1 - \alpha^2) d\theta}{1 + 2\alpha \cos \theta + \alpha^2} + \frac{1}{4s\pi} \int_0^\pi \frac{(\beta^2 - 1) d\theta}{1 - 2\beta \cos \theta + \beta^2}; \quad (23.)$$

in which, by (16.), $\alpha + \beta$ has been changed to $4s$. But, in general if $a^2 > b^2$,

$$\int_0^\pi \frac{d\theta}{a + b \cos \theta} = \frac{\pi}{\sqrt{a^2 - b^2}}, \quad (24.)$$

the radical being a positive quantity if a be such; therefore in the formula (23.),

$$\int_0^\pi \frac{(1 - \alpha^2) d\theta}{1 + 2\alpha \cos \theta + \alpha^2} = \pi, \quad (25.)$$

because, by (16.), α cannot exceed the limits 0 and $\frac{1}{2}$, s being supposed not to exceed the limits 0 and 1, so that $1 - \alpha^2$ is positive. On the other hand, β varies from 0 to 4, while s

varies from 0 to 1; and $\beta^2 - 1$ will be positive or negative, according as s is greater or less than the positive root of the equation

$$s^2 + s = \frac{1}{2}. \quad (26.)$$

Hence, in (23.), we must make

$$\int_0^\pi \frac{(\beta^2 - 1) d\theta}{1 - 2\beta \cos \theta + \beta^2} = \pi, \text{ or } = -\pi, \quad (27.)$$

according as

$$s > \text{ or } < \frac{\sqrt{3} - 1}{2}; \quad (28.)$$

and thus we find, under the same alternative,

$$F(p) = \frac{1}{4s}(1 \pm 1), \quad (29.)$$

that is,

$$F(p) = \frac{1}{2s}, \text{ or } = 0. \quad (30.)$$

But, by (17.),

$$s = \sqrt{\frac{1-p}{2}}; \quad (31.)$$

the function $F(p)$, or the definite integral (14.), receives therefore a sudden change of form when p , in varying from -1 to 1 , passes through the critical value

$$p = \sqrt{3} - 1; \quad (32.)$$

in such a manner that we have

$$F(p) = (2 - 2p)^{-\frac{1}{2}}, \quad \text{if } p < \sqrt{3} - 1; \quad (33.)$$

and, on the other hand,

$$F(p) = 0 \quad \text{if } p > \sqrt{3} - 1; \quad (34.)$$

For the critical value (32.) itself, we have

$$s = \frac{\sqrt{3} - 1}{2}, \quad \alpha = 2\sqrt{3} - 3, \quad \beta = 1, \quad (35.)$$

and the real part of (18.) becomes

$$\frac{1 - \alpha}{1 + 2\alpha \cos \theta + \alpha^2}; \quad (36.)$$

multiplying therefore by $d\theta$, integrating from $\theta = 0$ to $\theta = \pi$, and dividing by π , we find, by (25.) and (14.), this formula instead of (29.),

$$F(p) = \frac{1}{1 + \alpha} = \frac{1}{4s}, \quad (37.)$$

that is,

$$F(p) = \frac{1}{2}(2 - 2p)^{-\frac{1}{2}}, \quad \text{if } p = \sqrt{3} - 1; \quad (38.)$$

The value of the discontinuous function F is therefore, in this case, equal to the semisum of the two different values which that function receives, immediately before and after the variable p attains its critical value, as usually happens in other similar cases of discontinuity.

5. As verifications of the results (33.), (34.), we may consider the particular values $p = 0$, $p = 1$, which ought to give

$$F(0) = 2^{-\frac{1}{2}}, \quad F(1) = 0. \quad (39.)$$

Accordingly, when $p = 0$, the definitions (9.) and (14.) give

$$\vartheta = \sqrt{-1} \sin \theta, \quad (40.)$$

$$F(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta}{1 - \sqrt{-1} \sin \theta} = \frac{1}{\pi} \int_0^{\pi} \frac{d\theta}{1 + \sin \theta^2}; \quad (41.)$$

which easily gives, by (24.),

$$F(0) = \frac{2}{\pi} \int_0^{\pi} \frac{d\theta}{3 - \cos 2\theta} = \frac{1}{\pi} \int_0^{2\pi} \frac{d\theta}{3 - \cos \theta} = 2^{-\frac{1}{2}}. \quad (42.)$$

And when $p = 1$, we have

$$1 - \vartheta = -\frac{1}{2}(\cos \theta + \sqrt{-1} \sin \theta), \quad (43.)$$

$$\frac{1}{2\pi} \frac{d\theta}{1 - \vartheta} = -\pi^{-1}(\cos \theta - \sqrt{-1} \sin \theta) d\theta, \quad (44.)$$

of which the integral, taken from $\theta = -\pi$ to $\theta = \pi$, is $F(1) = 0$.

6. Let us consider now this other integral,

$$G(p) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\vartheta^n d\theta}{\vartheta - 1}. \quad (45.)$$

The expression (13.) gives

$$\Sigma_{(n)} {}_0^{n-1} P_n = F(p) + G(p); \quad (46.)$$

therefore, by (34.), we shall have

$$G(p) = \Sigma_{(n)} {}_0^{n-1} P_n, \quad \text{if } p > \sqrt{3} - 1. \quad (47.)$$

For instance, let $p = 1$; then multiplying the expression (44.) by

$$-\vartheta^n = -(1 + \frac{1}{2}e^{\theta\sqrt{-1}})^n, \quad (48.)$$

the only term which does not vanish when integrated is $\frac{1}{2}n\pi^{-1} d\theta$, and this term gives the result

$$G(1) = n, \quad (49.)$$

which evidently agrees with the formula (47.), because it is well known that

$$P_n = 1, \quad \text{when } p = 1, \quad (50.)$$

the series (2.) becoming then the development of $(1 - x)^{-1}$.

7. On the other hand, let p be $< \sqrt{3} - 1$; then, observing that, by (33.),

$$F(p) = (2 - 2p)^{-\frac{1}{2}} = \Sigma_{(n)} \infty P_n, \quad (51.)$$

we find, by the relation (46.) between the functions F and G ,

$$G(p) = -\Sigma_{(n)} \infty P_n = -(P_n + P_{n+1} + P_{n+2} + \dots). \quad (52.)$$

For instance, let $p = 0$; then, by (40.) and (45.),

$$G(0) = \frac{-(\sqrt{-1})^n}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta (\sin \theta)^n}{1 - \sqrt{-1} \sin \theta}; \quad (53.)$$

that is

$$G(0) = \frac{(-1)^{i+1}}{\pi} \int_0^{\pi} \int_0^{\pi} \frac{d\theta \sin \theta^{2i}}{1 + \sin \theta^2}; \quad (54.)$$

if n be either $= 2i - 1$, or $= 2i$. Now, when $p = 0$, P_n is the coefficient of x^n in the development of $(1 + x^2)^{-\frac{1}{2}}$; therefore,

$$P_{2i-1} = 0, \quad \text{when } p = 0, \quad (55.)$$

and, in the notation of factorials,

$$P_{2i} = [0]^{-i} [-\frac{1}{2}]^i = (-1)^i \pi^{-1} \int_0^{\pi} d\theta \sin \theta^{2i}; \quad (56.)$$

so that, by (54.),

$$G(0) = -(P_{2i} + P_{2i+2} + \dots), \quad (57.)$$

when $p = 0$, and when n is either $2i$ or $2i - 1$.

8. For the critical value $p = \sqrt{3} - 1$, we have, by (38.),

$$F(p) = \frac{1}{2} \Sigma_{(n)} \infty P_n; \quad (58.)$$

therefore, for the same value of p , by (46.),

$$\begin{aligned} G(p) &= \frac{1}{2} \Sigma_{(n)} \infty P_n - \frac{1}{2} \Sigma_{(n)} \infty P_n \\ &= \frac{1}{2} (P_0 + P_1 + \dots + P_{n-1} - P_n - P_{n+1} - \dots); \end{aligned} \quad (59.)$$

so that the discontinuous function G , like F , acquires, for the critical value of p , a value which is the semisum of those which it receives immediately before and afterwards.

9. We have seen that the sum of these two discontinuous integrals, F and G, is always equal to the sum of the first n terms of the series (12.), so that

$$F(p) + G(p) = P_0 + P_1 + \dots + P_{n-1}; \quad (60.)$$

and it may not be irrelevant to remark that this sum may be developed under this other form:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \frac{\vartheta^n - 1}{\vartheta - 1} = \Sigma_{(k)} \binom{n}{1} [n]^k [0]^{-k} Q_{k-1}; \quad (61.)$$

in which the factorial expression $[n]^k [0]^{-k}$ denotes the coefficient of x^k in the development of $(1+x)^n$; and

$$Q_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta (\vartheta - 1)^k. \quad (62.)$$

Thus

$$\left. \begin{aligned} P_0 &= Q_0; \\ P_0 + P_1 &= 2Q_0 + Q_1; \\ P_0 + P_1 + P_2 &= 3Q_0 + 3Q_1 + Q_2; \\ &\&c. \end{aligned} \right\} \quad (63.)$$

and consequently

$$\left. \begin{aligned} P_0 &= Q_0; \\ P_1 &= Q_0 + Q_1; \\ P_2 &= Q_0 + 2Q_1 + Q_2; \\ &\&c. \end{aligned} \right\} \quad (64.)$$

which last expressions, indeed, follow immediately from the formula (10.).

10. With respect to the calculation of $Q_0, Q_1, \&c.$ as functions of p , it may be noted, in conclusion, that, by (15.) and (62.), Q_k is the term independent of θ in the development of

$$2^{-k} (\alpha + e^{\theta\sqrt{-1}})^k (1 - \beta e^{-\theta\sqrt{-1}})^k; \quad (65.)$$

thus

$$\left. \begin{aligned} Q_0 &= 1, \\ Q_1 &= 2^{-1}(\alpha - \beta), \\ Q_2 &= 2^{-2}(\alpha^2 - 4\alpha\beta + \beta^2), \\ Q_3 &= 2^{-3}(\alpha^3 - 9\alpha^2\beta + 9\alpha\beta^2 - \beta^3), \\ &\&c. \end{aligned} \right\} \quad (66.)$$

in which the law of formation is evident. It remains to substitute for α, β their values (16.) as functions of s , and then to eliminate s^2 by (17.); and thus we find, for example,

$$\left. \begin{aligned} Q_1 &= p - 1, \\ Q_2 &= \frac{1}{2}(p - 1)(3p - 1); \\ Q_3 &= \frac{1}{2}(p - 1)^2(5p + 1); \\ Q_4 &= \frac{1}{8}(p - 1)^2(35p^2 - 10p - 13). \end{aligned} \right\} \quad (67.)$$

This, then, is at least one way, though perhaps not the easiest, of computing the initial values of the successive differences of the function P_n , that is, the quantities

$$\left. \begin{aligned} Q_0 &= \Delta^0 P_0 = P_0, \\ Q_1 &= \Delta^1 P_0 = P_1 - P_0, \\ Q_2 &= \Delta^2 P_0 = P_2 - 2P_1 + P_0, \\ &\quad \&c. \end{aligned} \right\} \quad (68.)$$

And we see that it is permitted to express generally those differences, as follows:

$$\Delta^k P_0 = s^k \sum_{(i)} \binom{k}{i} (-1)^i ([k]^i [0]^{-i})^2 (1+s)^i (1-s)^{k-i}; \quad (69.)$$

in which

$$s^2 = \frac{1}{2}(1-p). \quad (70.)$$

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Feb. 12, 1842.