# SECOND SUPPLEMENT TO AN ESSAY ON THE THEORY OF SYSTEMS OF RAYS

## By

## William Rowan Hamilton

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# NOTE ON THE TEXT

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*Second Supplement to an Essay on the Theory of Systems of Rays.* By WILLIAM R. HAMILTON, *Royal Astronomer of Ireland, &c. &c.*

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## INTRODUCTION.

The present Supplement contains the integration of some partial differential equations, to which I have been conducted by the view of mathematical optics, proposed in my former memoirs. According to that view, the geometrical properties of an optical system of rays may be deduced by analytic methods, from the form of one characteristic function; of which the partial differential coefficients of the first order, taken with respect to the three rectangular coordinates of any proposed point of the system, are, in the case of ordinary light, equal to the index of refraction of the medium, multiplied by the cosines of the angles which the ray passing through the point makes with the axes of coordinates: and as these cosines are connected by the known relation that the sum of their squares is unity, there results a corresponding connexion between the partial differential coefficients to which they are proportional. This connexion is expressed by an equation which it is interesting to study and to integrate, because it contains a general property of ordinary systems of rays, and because its integral is a general form for the characteristic function of such a system. The integral which I have given in the present memoir, is deduced from equations assigned in my former Supplement; an elimination which had been before supposed, being now effected, by the theorems which Laplace has established in the second Book of the *Mécanique Céleste*, for the development of functions into series. The development thus obtained, proceeds according to the ascending powers of the perpendicular distances of a variable point from the tangent planes of the two rectangular developable pencils which pass through an assumed ray of the system, and according to the descending powers of the distances of the projection of the variable point upon the assumed ray, from the points in which that ray touches the two caustic surfaces. In the case of rays contained in one plane, or symmetric about one axis, the partial differential equation takes simpler forms, of which I have assigned the integrals, and have given an example of their optical use, by briefly shewing their connexion with the longitudinal aberrations of curvature. I hope, in a future memoir, to point out other methods of integrating the general equation for the characteristic function of ordinary systems of rays, and other applications of the resulting expressions, to the solution of optical problems.

WILLIAM R. HAMILTON.

OBSERVATORY, *October* 1830.

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# SECOND SUPPLEMENT.

*Statement and Integration of the Partial Differential Equation, which determines the Characteristic Function of Ordinary Systems of Rays, produced by any Number of successive Reflexions or Refractions.*

1. Suppose that rays of a given colour diverge from a given luminous origin, and undergo any number of successive changes of direction, according to the known laws of ordinary reflexion and refraction, at surfaces having any given shapes and positions, and enclosing media of any given refractive indices. Let $\alpha$, $\beta$, $\gamma$, be the cosines of the angles which the direction of a final ray makes with three rectangular axes, and let $x$, $y$, $z$, be the three rectangular coordinates, referred to the same axes, of a point upon this final ray; then $\alpha$, $\beta$, $\gamma$, will in general be functions of $x$, $y$, $z$, such that if $\mu$ denote the refractive index of the final medium, for rays of the given colour, the expression

$$\mu(\alpha\,dx + \beta\,dy + \gamma\,dz)$$

is equal to the differential of a certain function $V$, of which I have shewn the existence and the meaning in former memoirs, and which I have called the *characteristic function* of the final system. The design of the present Supplement, is to point out some new properties and uses of this function, resulting from the partial differential equation

$$\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2 = \mu^2, \tag{A}$$

which we obtain by eliminating the three cosines $\alpha$, $\beta$, $\gamma$, between the three equations

$$\frac{dV}{dx} = \mu\alpha, \quad \frac{dV}{dy} = \mu\beta, \quad \frac{dV}{dz} = \mu\gamma, \tag{B}$$

by the help of the known relation

$$\alpha^2 + \beta^2 + \gamma^2 = 1.$$

2. The equation (A) is a particular case of a more general differential equation, for all optical systems of rays, ordinary or extraordinary, obtained by eliminating the same three cosines, $\alpha$, $\beta$, $\gamma$, by the same known relation between the three following equations, assigned in my former memoirs,

$$\frac{dV}{dx} = \frac{\delta v}{\delta\alpha}, \quad \frac{dV}{dy} = \frac{\delta v}{\delta\beta}, \quad \frac{dV}{dz} = \frac{\delta v}{\delta\gamma};$$

in which $V$ is the characteristic function of the system, and $v$ is a homogeneous function of $\alpha$, $\beta$, $\gamma$, of the first dimension, representing the velocity of the light, estimated on the hypothesis of emission, and differentiated as if $\alpha$, $\beta$, $\gamma$, were three independent variables. And the integral of (A), is a particular case of a more general integral, extending to all optical systems of straight rays, and consisting of the following combination of equations, assigned in my former Supplement:

$$\begin{aligned}
W + V &= x\frac{\delta v}{\delta\alpha} + y\frac{\delta v}{\delta\beta} + z\frac{\delta v}{\delta\gamma},\\
\frac{\delta W}{\delta\alpha} &= x\frac{\delta^2 v}{\delta\alpha^2} + y\frac{\delta^2 v}{\delta\alpha\,\delta\beta} + z\frac{\delta^2 v}{\delta\alpha\,\delta\gamma},\\
\frac{\delta W}{\delta\beta} &= x\frac{\delta^2 v}{\delta\alpha\,\delta\beta} + y\frac{\delta^2 v}{\delta\beta^2} + z\frac{\delta^2 v}{\delta\beta\,\delta\gamma},\\
\frac{\delta W}{\delta\gamma} &= x\frac{\delta^2 v}{\delta\alpha\,\delta\gamma} + y\frac{\delta^2 v}{\delta\beta\,\delta\gamma} + z\frac{\delta^2 v}{\delta\gamma^2};
\end{aligned}$$

between which the three quantities $\alpha$, $\beta$, $\gamma$, are to be eliminated; $W$ being an arbitrary but homogeneous function of these three quantities, of the dimension zero; and the partial differential coefficients in which the sign $\delta$ occurs, being formed by differentiating the homogeneous functions $W$, $v$, as if $\alpha$, $\beta$, $\gamma$, were three independent variables. In applying these general results to ordinary systems of rays, we are to put

$$v = \mu(\alpha^2 + \beta^2 + \gamma^2)^{\frac{1}{2}};$$

$$\frac{\delta v}{\delta\alpha} = \frac{\mu^2\alpha}{v}, \quad \frac{\delta v}{\delta\beta} = \frac{\mu^2\beta}{v}, \quad \frac{\delta v}{\delta\gamma} = \frac{\mu^2\gamma}{v};$$

$$\frac{\delta^2 v}{\delta\alpha^2} = \frac{\mu^2}{v^3}(v^2 - \mu^2\alpha^2), \quad \frac{\delta^2 v}{\delta\beta^2} = \frac{\mu^2}{v^3}(v^2 - \mu^2\beta^2), \quad \frac{\delta^2 v}{\delta\gamma^2} = \frac{\mu^2}{v^3}(v^2 - \mu^2\gamma^2);$$

$$\frac{\delta^2 v}{\delta\alpha\,\delta\beta} = -\frac{\mu^4\alpha\beta}{v^3}, \quad \frac{\delta^2 v}{\delta\beta\,\delta\gamma} = -\frac{\mu^4\beta\gamma}{v^3}, \quad \frac{\delta^2 v}{\delta\gamma\,\delta\alpha} = -\frac{\mu^4\gamma\alpha}{v^3}:$$

or, (making after the differentiations $\alpha^2 + \beta^2 + \gamma^2 = 1$,)

$$v = \mu, \quad \frac{\delta v}{\delta\alpha} = \mu\alpha, \quad \frac{\delta v}{\delta\beta} = \mu\beta, \quad \frac{\delta v}{\delta\gamma} = \mu\gamma,$$

$$\frac{\delta^2 v}{\delta\alpha^2} = \mu(1 - \alpha^2), \quad \frac{\delta^2 v}{\delta\beta^2} = \mu(1 - \beta^2), \quad \frac{\delta^2 v}{\delta\gamma^2} = \mu(1 - \gamma^2),$$

$$\frac{\delta^2 v}{\delta\alpha\,\delta\beta} = -\mu\alpha\beta, \quad \frac{\delta^2 v}{\delta\beta\,\delta\gamma} = -\mu\beta\gamma, \quad \frac{\delta^2 v}{\delta\gamma\,\delta\alpha} = -\mu\gamma\alpha;$$

and therefore,

$$\left.\begin{aligned}
W + V &= \mu(\alpha x + \beta y + \gamma z),\\
\frac{\delta W}{\delta\alpha} &= \mu x - \mu\alpha(\alpha x + \beta y + \gamma z),\\
\frac{\delta W}{\delta\beta} &= \mu y - \mu\beta(\alpha x + \beta y + \gamma z),\\
\frac{\delta W}{\delta\gamma} &= \mu z - \mu\gamma(\alpha x + \beta y + \gamma z).
\end{aligned}\right\} \tag{C}$$

This system of equations (C) is one form for the integral of the partial differential equation (A); the quantities $\alpha$, $\beta$, $\gamma$, being supposed to be eliminated, and $W$ being an arbitrary function of these quantities, of the kind already mentioned.

3. The system of equations (C) may be transformed into the following:

$$\mu x = \frac{dU}{d\alpha}, \quad \mu y = \frac{dU}{d\beta}, \quad V + U = \alpha\frac{dU}{d\alpha} + \beta\frac{dU}{d\beta}; \tag{D}$$

in which $U$ is a function of the three independent variables $\alpha$, $\beta$, $z$, obtained from the function $W$ by putting

$$U = W - \mu\gamma z, \tag{E}$$

and by considering $\gamma$ as a function of $\alpha$, $\beta$. Let us now proceed to eliminate $\alpha$, $\beta$, between the three equations (D), by the theorems which Laplace has given in the second Book of the *Mécanique Céleste*, for the development of functions into series.

This elimination may be simplified by a proper choice of the coordinates. The rays of an ordinary system being perpendicular to the surfaces which have for equation

$$V = \text{const.},$$

compose in general two series of rectangular developable pencils, and are tangents to two caustic surfaces. Let us therefore denote by $x_{\prime}$, $y_{\prime}$, $z_{\prime}$, three rectangular coordinates so chosen that the axis of $z_{\prime}$ coincides with some given ray, and that the planes of $x_{\prime}z_{\prime}$ and $y_{\prime}z_{\prime}$ are the tangent planes of the two developable pencils to which that ray belongs; and let $\alpha$ $\beta$ $\gamma$ denote, for any proposed ray of the system, the cosines of the angles which the ray makes with the axes of $x_{\prime}$ $y_{\prime}$ $z_{\prime}$. The equations (A) (B) (C) (D) (E) will apply to the coordinates thus chosen, by simply changing $x$ $y$ $z$ to $x_{\prime}$ $y_{\prime}$ $z_{\prime}$; and by changing $\gamma$ to its value

$$\gamma = \sqrt{1 - \alpha^2 - \beta^2} = 1 - \frac{\alpha^2 + \beta^2}{2} - \gamma^{(4)} - \gamma^{(6)} - \&\text{c.},$$

in which

$$\gamma^{(2i+4)} = \frac{1\,.\,3\,.\,5\,\ldots\,(2i+1)}{2\,.\,4\,.\,6\,\ldots\,(2i+2)}\,\frac{(\alpha^2 + \beta^2)^{i+2}}{2i+4},$$

the function $W$ will in general admit of being thus developed,

$$W = \mu W^{(0)} + \frac{\mu}{2}(A\alpha^2 + B\beta^2) + \mu W^{(3)} + \mu W^{(4)} + \&\text{c.}, \tag{F}$$

$W^{(0)}$, $A$, $B$, being constants, and $W^{(3)}$, $W^{(4)}$, $W^{(i)}$, being rational homogeneous functions of the two small variables $\alpha$, $\beta$, of the dimensions 3, 4, $i$, respectively. The constants $A$, $B$, are here the distances upon the ray, from the point in which it touches the two caustic surfaces, to the origin of the coordinates $x_{\prime}$ $y_{\prime}$ $z_{\prime}$; and the terms proportional to $\alpha$, $\beta$, $\alpha\beta$, disappear from the development of $W$, by the choice which we have made of these coordinates, and by the principles of the former Supplement. In this manner the function $U$ becomes

$$U = \mu W^{(0)} - \mu z_{\prime} + \frac{\mu}{2}\{(z_{\prime} + A)\alpha^2 + (z_{\prime} + B)\beta^2\} + \mu U^{(3)} + \mu U^{(4)} + \&\text{c.}, \tag{G}$$

in which

$$U^{(2i+3)} = W^{(2i+3)}; \quad U^{(2i+4)} = W^{(2i+4)} + z_{\prime}\gamma^{(2i+4)};$$

and the two first of the equations (D) become

$$\alpha = \alpha_{\prime} + (z_{\prime} + A)^{-1}\frac{d\phi}{d\alpha}; \quad \beta = \beta_{\prime} + (z_{\prime} + B)^{-1}\frac{d\phi}{d\beta}, \tag{H}$$

if we put for abridgment

$$\alpha_{\prime} = \frac{x_{\prime}}{z_{\prime} + A}, \quad \beta_{\prime} = \frac{y_{\prime}}{z_{\prime} + B}, \quad \phi = -(U^{(3)} + U^{(4)} + \&\text{c.}). \tag{I}$$

On account of the smallness of $\frac{d\phi}{d\alpha}, \frac{d\phi}{d\beta}$, the quantities $\alpha_{\prime}$, $\beta_{\prime}$, are approximate values of $\alpha$, $\beta$; and to develope $\alpha$, $\beta$, themselves, or any function of them, $F(\alpha, \beta)$, in a series of ascending powers of these approximate values, we have, by the theorems of Laplace before referred to,

$$F(\alpha,\beta) = F_{\prime} + \Sigma_{(n)}{}_0^\infty \left\{ \frac{\frac{d^n}{d\alpha_{\prime}^n}\left(\frac{dF_{\prime}}{d\alpha_{\prime}}\left(\frac{d\phi_{\prime}}{d\alpha_{\prime}}\right)^{n+1}\right)}{[n+1]^{n+1}(z_{\prime}+A)^{n+1}} + \frac{\frac{d^n}{d\beta_{\prime}^n}\left(\frac{dF_{\prime}}{d\beta_{\prime}}\left(\frac{d\phi_{\prime}}{d\beta_{\prime}}\right)^{n+1}\right)}{[n+1]^{n+1}(z_{\prime}+B)^{n+1}} \right\}$$
$$+ \Sigma_{(n,n')}{}_{0,\ 0}^{\infty,\infty} \frac{\frac{d^{n+n'}}{d\alpha_{\prime}^n\, d\beta_{\prime}^{n'}} \left\{ \begin{array}{l} \frac{d^2F_{\prime}}{d\alpha_{\prime}\, d\beta_{\prime}} \left(\frac{d\phi_{\prime}}{d\alpha_{\prime}}\right)^{n+1} \left(\frac{d\phi_{\prime}}{d\beta_{\prime}}\right)^{n'+1} \\ + \frac{dF_{\prime}}{d\alpha_{\prime}} \left(\frac{d\phi_{\prime}}{d\beta_{\prime}}\right)^{n'+1} \frac{d}{d\beta_{\prime}} \left(\frac{d\phi_{\prime}}{d\alpha_{\prime}}\right)^{n+1} \\ + \frac{dF_{\prime}}{d\beta_{\prime}} \left(\frac{d\phi_{\prime}}{d\alpha_{\prime}}\right)^{n+1} \frac{d}{d\alpha_{\prime}} \left(\frac{d\phi_{\prime}}{d\beta_{\prime}}\right)^{n'+1} \end{array} \right\}}{[n+1]^{n+1}[n'+1]^{n'+1}(z_{\prime}+A)^{n+1}(z_{\prime}+B)^{n'+1}}, \tag{K}$$

the functions $F_{\prime}$, $\phi_{\prime}$, being formed from $F$, $\phi$, by changing $\alpha$, $\beta$, to $\alpha_{\prime}$, $\beta_{\prime}$, and $[n+1]^{n+1}$, $[n'+1]^{n'+1}$, being known factorial symbols; we have therefore,

$$
\left.\begin{aligned}
\alpha &= \alpha_{\prime} + \Sigma_{(n)}{}_{0}^{\infty} \frac{\dfrac{d^n}{d\alpha_{\prime}^n} \cdot \left(\dfrac{d\phi_{\prime}}{d\alpha_{\prime}}\right)^{n+1}}{[n+1]^{n+1}(z_{\prime}+A)^{n+1}} \\
&\quad + \Sigma_{(n,n')}{}_{0,\ 0}^{\infty,\infty} \frac{\dfrac{d^{n+n'}}{d\alpha_{\prime}^n\, d\beta_{\prime}^{n'}} \left(\dfrac{d^2\phi_{\prime}}{d\alpha_{\prime}\, d\beta_{\prime}} \left(\dfrac{d\phi_{\prime}}{d\alpha_{\prime}}\right)^{n} \left(\dfrac{d\phi_{\prime}}{d\beta_{\prime}}\right)^{n'+1}\right)}{[n]^n[n'+1]^{n'+1}(z_{\prime}+A)^{n+1}(z_{\prime}+B)^{n'+1}}; \\
\beta &= \beta_{\prime} + \Sigma_{(n)}{}_{0}^{\infty} \frac{\dfrac{d^n}{d\beta_{\prime}^n} \cdot \left(\dfrac{d\phi_{\prime}}{d\beta_{\prime}}\right)^{n+1}}{[n+1]^{n+1}(z_{\prime}+B)^{n+1}} \\
&\quad + \Sigma_{(n,n')}{}_{0,\ 0}^{\infty,\infty} \frac{\dfrac{d^{n+n'}}{d\alpha_{\prime}^n\, d\beta_{\prime}^{n'}} \left(\dfrac{d^2\phi_{\prime}}{d\alpha_{\prime}\, d\beta_{\prime}} \left(\dfrac{d\phi_{\prime}}{d\alpha_{\prime}}\right)^{n+1} \left(\dfrac{d\phi_{\prime}}{d\beta_{\prime}}\right)^{n'}\right)}{[n+1]^{n+1}[n']^{n'}(z_{\prime}+A)^{n+1}(z_{\prime}+B)^{n'+1}}.
\end{aligned}\right\} \qquad \text{(L)}
$$

Now, if we differentiate $V$ as a function of the three independent variables $\alpha_{\prime}$, $\beta_{\prime}$, $z_{\prime}$, we have by (B) and (I),

$$
\frac{dV}{d\alpha_{\prime}} = \mu\alpha(z_{\prime}+A), \quad \frac{dV}{d\beta_{\prime}} = \mu\beta(z_{\prime}+B), \quad \frac{dV}{dz_{\prime}} = \mu(\alpha\alpha_{\prime}+\beta\beta_{\prime}+\gamma); \qquad \text{(M)}
$$

we have also $V = \mu z_{\prime} - \mu W^{(0)}$, when $\alpha_{\prime}$, $\beta_{\prime}$, vanish; and therefore,

$$
V = \mu z_{\prime} - \mu W^{(0)} + \mu \int \{(z_{\prime}+A)\alpha\, d\alpha_{\prime} + (z_{\prime}+B)\beta\, d\beta_{\prime}\}, \qquad \text{(N)}
$$

$z_{\prime}$ being considered as constant in the integration, and the integral being so determined as to vanish with $\alpha_{\prime}$, $\beta_{\prime}$. Substituting in this expression (N), the developments of $\alpha$, $\beta$, and performing the integration, we find the following development for $\dfrac{V}{\mu}$,

$$
\begin{aligned}
\frac{V}{\mu} &= z_{\prime} - W^{(0)} + \tfrac{1}{2}\{(z_{\prime}+A)\alpha_{\prime}^2 + (z_{\prime}+B)\beta_{\prime}^2\} \\
&\quad + \phi_{\prime} + \Sigma_{(n)}{}_{0}^{\infty} \left\{ \frac{\dfrac{d^n}{d\alpha_{\prime}^n} \cdot \left(\dfrac{d\phi_{\prime}}{d\alpha_{\prime}}\right)^{n+2}}{[n+2]^{n+2}(z_{\prime}+A)^{n+1}} + \frac{\dfrac{d^n}{d\beta_{\prime}^n} \cdot \left(\dfrac{d\phi_{\prime}}{d\beta_{\prime}}\right)^{n+2}}{[n+2]^{n+2}(z_{\prime}+B)^{n+1}} \right\} \\
&\quad + \Sigma_{(n,n')}{}_{0,\ 0}^{\infty,\infty} \frac{\dfrac{d^{n+n'}}{d\alpha_{\prime}^n\, d\beta_{\prime}^{n'}} \left(\dfrac{d^2\phi_{\prime}}{d\alpha_{\prime}\, d\beta_{\prime}} \left(\dfrac{d\phi_{\prime}}{d\alpha_{\prime}}\right)^{n+1} \left(\dfrac{d\phi_{\prime}}{d\beta_{\prime}}\right)^{n'+1}\right)}{[n+1]^{n+1}[n'+1]^{n'+1}(z_{\prime}+A)^{n+1}(z_{\prime}+B)^{n'+1}}; \qquad \text{(O)}
\end{aligned}
$$

which is another form for the integral of the partial differential equation (A), obtained from the elimination (D). And if we wish to introduce any other rectangular coordinates $x$, $y$, $z$, into the expression of this integral (O), instead of $x_\prime$, $y_\prime$, $z_\prime$, we may do so by the known methods, by putting

$$\left.\begin{aligned} x_\prime &= (x - x_{\prime\prime}) \cos. xx_\prime + (y - y_{\prime\prime}) \cos. yx_\prime + (z - z_{\prime\prime}) \cos. zx_\prime, \\ y_\prime &= (x - x_{\prime\prime}) \cos. xy_\prime + (y - y_{\prime\prime}) \cos. yy_\prime + (z - z_{\prime\prime}) \cos. zy_\prime, \\ z_\prime &= (x - x_{\prime\prime}) \cos. xz_\prime + (y - y_{\prime\prime}) \cos. yz_\prime + (z - z_{\prime\prime}) \cos. zz_\prime, \end{aligned}\right\} \tag{P}$$

$x_{\prime\prime}$, $y_{\prime\prime}$, $z_{\prime\prime}$, being the values of $x$, $y$, $z$, that belong to the point upon the ray which had been taken for origin.

*Verifications of the foregoing Developments.*

4. We may verify the form (O) which we have thus found for the integral of (A), by the folowing condition, resulting from (M),

$$\frac{d}{dz_\prime} \cdot \frac{V}{\mu} - \frac{\alpha_\prime}{z_\prime + A} \frac{d}{d\alpha_\prime} \cdot \frac{V}{\mu} - \frac{\beta_\prime}{z_\prime + B} \frac{d}{d\beta_\prime} \cdot \frac{V}{\mu} = \sqrt{1 - \alpha^2 - \beta^2}, \tag{Q}$$

of which each member is an expression for the cosine $\gamma$ of the small angle which a near ray makes with the ray that we have taken for the axis of $z_\prime$. The condition (Q) may be put under the form

$$\frac{d}{dz_\prime} \cdot \frac{V}{\mu} - (\alpha\alpha_\prime + \beta\beta_\prime) = \sqrt{1 - \alpha^2 - \beta^2} \tag{R}$$

in which, by (O),

$$\begin{aligned} \frac{d}{dz_\prime} \cdot \frac{V}{\mu} = 1 + \frac{\alpha_\prime^2 + \beta_\prime^2}{2} + \frac{d\phi_\prime}{dz_\prime} \\ + \Sigma_{(n)}\,{}_0^\infty \left\{ \frac{\dfrac{d^n}{d\alpha_\prime^n} \left( \left(\dfrac{d\phi_\prime}{d\alpha_\prime}\right)^{n+1} \dfrac{d^2\phi_\prime}{d\alpha_\prime\, dz_\prime} \right)}{[n+1]^{n+1} (z_\prime + A)^{n+1}} + \frac{\dfrac{d^n}{d\beta_\prime^n} \left( \left(\dfrac{d\phi_\prime}{d\beta_\prime}\right)^{n+1} \dfrac{d^2\phi_\prime}{d\beta_\prime\, dz_\prime} \right)}{[n+1]^{n+1} (z_\prime + B)^{n+1}} \right\} \\ - \Sigma_{(n)}\,{}_0^\infty \left\{ \frac{\dfrac{d^n}{d\alpha_\prime^n} \cdot \left(\dfrac{d\phi_\prime}{d\alpha_\prime}\right)^{n+2}}{[n]^n (n+2) (z_\prime + A)^{n+2}} + \frac{\dfrac{d^n}{d\beta_\prime^n} \cdot \left(\dfrac{d\phi_\prime}{d\beta_\prime}\right)^{n+2}}{[n]^n (n+2) (z_\prime + B)^{n+2}} \right\} \\ + \Sigma_{(n,n')}\,{}_{0,\ 0}^{\infty,\infty} \frac{\dfrac{d^{n+n'+1}}{d\alpha_\prime^n\, d\beta_\prime^{n'}\, dz_\prime} \left( \dfrac{d^2\phi_\prime}{d\alpha_\prime\, d\beta_\prime} \left(\dfrac{d\phi_\prime}{d\alpha_\prime}\right)^{n+1} \left(\dfrac{d\phi_\prime}{d\beta_\prime}\right)^{n'+1} \right)}{[n+1]^{n+1} [n'+1]^{n'+1} (z_\prime + A)^{n+1} (z_\prime + B)^{n'+1}} \\ - \Sigma_{(n,n')}\,{}_{0,\ 0}^{\infty,\infty} \left( \frac{z_\prime + A}{n+1} + \frac{z_\prime + B}{n'+1} \right) \frac{\dfrac{d^{n+n'}}{d\alpha_\prime^n\, d\beta_\prime^{n'}} \left( \dfrac{d^2\phi_\prime}{d\alpha_\prime\, d\beta_\prime} \left(\dfrac{d\phi_\prime}{d\alpha_\prime}\right)^{n+1} \left(\dfrac{d\phi_\prime}{d\beta_\prime}\right)^{n'+1} \right)}{[n]^n [n']^{n'} (z_\prime + A)^{n+2} (z_\prime + B)^{n'+2}}; \end{aligned} \tag{S}$$

and, by (L),

$$\alpha\alpha_{,} + \beta\beta_{,} = \alpha_{,}^2 + \beta_{,}^2 + \Sigma_{(n)}{}_0^\infty \left\{ \frac{\alpha_{,} \cdot \dfrac{d^n}{d\alpha_{,}^n} \left(\dfrac{d\phi_{,}}{d\alpha_{,}}\right)^{n+1}}{[n+1]^{n+1}(z_{,}+A)^{n+1}} + \frac{\beta_{,} \cdot \dfrac{d^n}{d\beta_{,}^n} \left(\dfrac{d\phi_{,}}{d\beta_{,}}\right)^{n+1}}{[n+1]^{n+1}(z_{,}+B)^{n+1}} \right\}$$

$$+ \alpha_{,} \Sigma_{(n,n')}{}_{0,\ 0}^{\infty,\infty} \frac{\dfrac{d^{n+n'}}{d\alpha_{,}^n\, d\beta_{,}^{n'}} \left( \dfrac{d^2\phi_{,}}{d\alpha_{,}\, d\beta_{,}} \left(\dfrac{d\phi_{,}}{d\alpha_{,}}\right)^n \left(\dfrac{d\phi_{,}}{d\beta_{,}}\right)^{n'+1} \right)}{[n]^n [n'+1]^{n'+1} (z_{,}+A)^{n+1} (z_{,}+B)^{n'+1}}$$

$$+ \beta_{,} \Sigma_{(n,n')}{}_{0,\ 0}^{\infty,\infty} \frac{\dfrac{d^{n+n'}}{d\alpha_{,}^n\, d\beta_{,}^{n'}} \left( \dfrac{d^2\phi_{,}}{d\alpha_{,}\, d\beta_{,}} \left(\dfrac{d\phi_{,}}{d\alpha_{,}}\right)^{n+1} \left(\dfrac{d\phi_{,}}{d\beta_{,}}\right)^{n'} \right)}{[n+1]^{n+1} [n']^{n'} (z_{,}+A)^{n+1} (z_{,}+B)^{n'+1}}; \tag{T}$$

while the development of

$$\sqrt{1-\alpha^2-\beta^2}$$

may be deduced from the general formula (K) by changing

$$F(\alpha,\beta) \quad \text{to} \quad \gamma = \sqrt{1-\alpha^2-\beta^2},$$

$$F_{,} \quad \text{to} \quad \gamma_{,} = \sqrt{1-\alpha_{,}^2-\beta_{,}^2}.$$

To compare these several developments, and to examine whether they satisfy the condition (R), we are to observe, that from the nature of the function $\phi$, we have by the foregoing number,

$$\left.\begin{aligned} \frac{d\phi_{,}}{dz_{,}} = -(\gamma_{,}^{(4)} + \gamma_{,}^{(6)} + \&\text{c.}) = \gamma_1 - 1 + \frac{\alpha_{,}^2+\beta_{,}^2}{2}; \\ \frac{d^2\phi_{,}}{d\alpha_{,}\, dz_{,}} = \alpha_{,} + \frac{d\gamma_{,}}{d\alpha_{,}}; \quad \frac{d^2\phi_{,}}{d\beta_{,}\, dz_{,}} = \beta_{,} + \frac{d\gamma_{,}}{d\beta_{,}}; \quad \frac{d^3\phi_{,}}{d\alpha_{,}\, d\beta_{,}\, dz_{,}} = \frac{d^2\gamma_{,}}{d\alpha_{,}\, d\beta_{,}}; \end{aligned}\right\} \tag{U}$$

and therefore

$$\begin{aligned} \frac{d}{dz_{,}} \left( \frac{d^2\phi_{,}}{d\alpha_{,}\, d\beta_{,}} \left(\frac{d\phi_{,}}{d\alpha_{,}}\right)^{n+1} \left(\frac{d\phi_{,}}{d\beta_{,}}\right)^{n'+1} \right) = {} & \frac{d^2\gamma_{,}}{d\alpha_{,}\, d\beta_{,}} \left(\frac{d\phi_{,}}{d\alpha_{,}}\right)^{n+1} \left(\frac{d\phi_{,}}{d\beta_{,}}\right)^{n'+1} \\ & + \left(\alpha_{,} + \frac{d\gamma_{,}}{d\alpha_{,}}\right) \left(\frac{d\phi_{,}}{d\beta_{,}}\right)^{n'+1} \frac{d}{d\beta_{,}} \cdot \left(\frac{d\phi_{,}}{d\alpha_{,}}\right)^{n+1} \\ & + \left(\beta_{,} + \frac{d\gamma_{,}}{d\beta_{,}}\right) \left(\frac{d\phi_{,}}{d\alpha_{,}}\right)^{n+1} \frac{d}{d\alpha_{,}} \cdot \left(\frac{d\phi_{,}}{d\beta_{,}}\right)^{n'+1}; \end{aligned}$$

by which means the difference of the developments (S) and (T) becomes

$$\frac{d}{dz_\prime}\cdot\frac{V}{\mu}-(\alpha\alpha_\prime+\beta\beta_\prime)=\gamma_\prime+\Sigma_{(n)}{}_0^\infty\left\{\frac{\dfrac{d^n}{d\alpha_\prime^n}\left(\left(\dfrac{d\phi_\prime}{d\alpha_\prime}\right)^{n+1}\dfrac{d\gamma_\prime}{d\alpha_\prime}\right)}{[n+1]^{n+1}(z_\prime+A)^{n+1}}+\frac{\dfrac{d^n}{d\beta_\prime^n}\left(\left(\dfrac{d\phi_\prime}{d\beta_\prime}\right)^{n+1}\dfrac{d\gamma_\prime}{d\beta_\prime}\right)}{[n+1]^{n+1}(z_\prime+B)^{n+1}}\right\}$$

$$+\Sigma_{(n,n')}{}_{0,\ 0}^{\infty,\infty}\frac{\dfrac{d^{n+n'}}{d\alpha_\prime^n\,d\beta_\prime^{n'}}\left\{\begin{array}{c}\dfrac{d^2\gamma_\prime}{d\alpha_\prime\,d\beta_\prime}\left(\dfrac{d\phi_\prime}{d\alpha_\prime}\right)^{n+1}\left(\dfrac{d\phi_\prime}{d\beta_\prime}\right)^{n'+1}\\ +\dfrac{d\gamma_\prime}{d\alpha_\prime}\left(\dfrac{d\phi_\prime}{d\beta_\prime}\right)^{n'+1}\dfrac{d}{d\beta_\prime}\cdot\left(\dfrac{d\phi_\prime}{d\alpha_\prime}\right)^{n+1}\\ +\dfrac{d\gamma_\prime}{d\beta_\prime}\left(\dfrac{d\phi_\prime}{d\alpha_\prime}\right)^{n+1}\dfrac{d}{d\alpha_\prime}\cdot\left(\dfrac{d\phi_\prime}{d\beta_\prime}\right)^{n'+1}\end{array}\right\}}{[n+1]^{n+1}[n'+1]^{n'+1}(z_\prime+A)^{n+1}(z_\prime+B)^{n'+1}},\tag{V}$$

and the series in this second member being exactly that which would result in the development of

$$\gamma=\sqrt{1-\alpha^2-\beta^2},$$

from the formula (K), we see that the condition (Q) or (R) is satisfied, and the sought verification is obtained.

Another verification of the foregoing developments may be obtained by applying the general expression in series (K), for any function $F$ of the cosines $\alpha$, $\beta$, to the case where this function is $=\dfrac{d\phi}{d\alpha}$. We find, first

$$\frac{d\phi}{d\alpha}=\frac{d\phi_\prime}{d\alpha_\prime}+\Sigma_{(n)}{}_0^\infty\left\{\frac{\dfrac{d^n}{d\alpha_\prime^n}\left(\left(\dfrac{d\phi_\prime}{d\alpha_\prime}\right)^{n+1}\dfrac{d^2\phi_\prime}{d\alpha_\prime^2}\right)}{[n+1]^{n+1}(z_\prime+A)^{n+1}}+\frac{\dfrac{d^n}{d\beta_\prime^n}\left(\left(\dfrac{d\phi_\prime}{d\beta_\prime}\right)^{n+1}\dfrac{d^2\phi_\prime}{d\alpha_\prime\,d\beta_\prime}\right)}{[n+1]^{n+1}(z_\prime+B)^{n+1}}\right\}$$

$$+\Sigma_{(n,n')}{}_{0,\ 0}^{\infty,\infty}\frac{\dfrac{d^{n+n'}}{d\alpha_\prime^n\,d\beta_\prime^{n'}}\left\{\begin{array}{c}\dfrac{d^3\phi_\prime}{d\alpha_\prime^2\,d\beta_\prime}\left(\dfrac{d\phi_\prime}{d\alpha_\prime}\right)^{n+1}\left(\dfrac{d\phi_\prime}{d\beta_\prime}\right)^{n'+1}\\ +\dfrac{d^2\phi_\prime}{d\alpha_\prime^2}\left(\dfrac{d\phi_\prime}{d\beta_\prime}\right)^{n'+1}\dfrac{d}{d\beta_\prime}\cdot\left(\dfrac{d\phi_\prime}{d\alpha_\prime}\right)^{n+1}\\ +\dfrac{d^2\phi_\prime}{d\alpha_\prime\,d\beta_\prime}\left(\dfrac{d\phi_\prime}{d\alpha_\prime}\right)^{n+1}\dfrac{d}{d\alpha_\prime}\cdot\left(\dfrac{d\phi_\prime}{d\beta_\prime}\right)^{n'+1}\end{array}\right\}}{[n+1]^{n+1}[n'+1]^{n'+1}(z_\prime+A)^{n+1}(z_\prime+B)^{n'+1}},\tag{W}$$

which may be put under the form

$$\frac{d\phi}{d\alpha} = \Sigma_{(n)}{}_0^\infty \frac{\dfrac{d^n}{d\alpha_,^n} \cdot \left(\dfrac{d\phi_,}{d\alpha_,}\right)^{n+1}}{[n+1]^{n+1}(z_, + A)^n} + \Sigma_{(n,n')}{}_{0,\ 0}^{\infty,\infty} \frac{\dfrac{d^{n+n'}}{d\alpha_,^n\, d\beta_,^{n'}} \left(\dfrac{d^2\phi_,}{d\alpha_,\, d\beta_,} \left(\dfrac{d\phi_,}{d\alpha_,}\right)^n \left(\dfrac{d\phi_,}{d\beta_,}\right)^{n'+1}\right)}{[n]^n[n'+1]^{n'+1}(z_, + A)^n(z_, + B)^{n'+1}}, \tag{X}$$

that is, by (L),

$$\frac{d\phi}{d\alpha} = (z_, + A)(\alpha - \alpha_,), \tag{Y}$$

which agrees with the conditions (H). A similar verification may be obtained by the same conditions (H), by considering the development of $\dfrac{d\phi}{d\beta}$.

Finally, we may observe that the condition

$$\frac{V}{\mu} = \alpha x_, + \beta y_, + \gamma z_, - \frac{W}{\mu} = \alpha x_, + \beta y_, - \frac{U}{\mu} \tag{Z}$$

becomes, by (G) and (I),

$$\frac{V}{\mu} = z_, - W^{(0)} + (z_, + A)\left(\alpha\alpha_, - \frac{\alpha^2}{2}\right) + (z_, + B)\left(\beta\beta_, - \frac{\beta^2}{2}\right) + \phi; \tag{A$'$}$$

in which, by (K) and (L),

$$\alpha\alpha_, - \frac{\alpha^2}{2} = \frac{\alpha_,^2}{2} - \Sigma_{(n)}{}_0^\infty \frac{\dfrac{n+1}{n+2}\dfrac{d^n}{d\alpha_,^n}\left(\dfrac{d\phi_,}{d\alpha_,}\right)^{n+2}}{[n+1]^{n+1}(z_, + A)^{n+2}}$$
$$- \Sigma_{(n,n')}{}_{0,\ 0}^{\infty,\infty} \frac{\dfrac{d^{n+n'}}{d\alpha_,^n\, d\beta_,^{n'}} \left(\dfrac{d^2\phi_,}{d\alpha_,\, d\beta_,} \left(\dfrac{d\phi_,}{d\alpha_,}\right)^{n+1} \left(\dfrac{d\phi_,}{d\beta_,}\right)^{n'+1}\right)}{[n]^n[n'+1]^{n'+1}(z_, + A)^{n+2}(z_, + B)^{n'+1}},$$

$$\beta\beta_, - \frac{\beta^2}{2} = \frac{\beta_,^2}{2} - \Sigma_{(n)}{}_0^\infty \frac{\dfrac{n+1}{n+2}\dfrac{d^n}{d\beta_,^n}\left(\dfrac{d\phi_,}{d\beta_,}\right)^{n+2}}{[n+1]^{n+1}(z_, + B)^{n+2}}$$
$$- \Sigma_{(n,n')}{}_{0,\ 0}^{\infty,\infty} \frac{\dfrac{d^{n+n'}}{d\alpha_,^n\, d\beta_,^{n'}} \left(\dfrac{d^2\phi_,}{d\alpha_,\, d\beta_,} \left(\dfrac{d\phi_,}{d\alpha_,}\right)^{n+1} \left(\dfrac{d\phi_,}{d\beta_,}\right)^{n'+1}\right)}{[n+1]^{n+1}[n']^{n'}(z_, + A)^{n+1}(z_, + B)^{n'+2}},$$

$$\phi = \phi_, + \Sigma_{(n)}{}_0^\infty \left\{ \frac{\dfrac{d^n}{d\alpha_,^n}\left(\dfrac{d\phi_,}{d\alpha_,}\right)^{n+2}}{[n+1]^{n+1}(z_, + A)^{n+1}} + \frac{\dfrac{d^n}{d\beta_,^n}\left(\dfrac{d\phi_,}{d\beta_,}\right)^{n+2}}{[n+1]^{n+1}(z_, + B)^{n+1}} \right\}$$
$$+ \Sigma_{(n,n')}{}_{0,\ 0}^{\infty,\infty} \frac{(n+n'+3)\dfrac{d^{n+n'}}{d\alpha_,^n\, d\beta_,^{n'}} \left(\dfrac{d^2\phi_,}{d\alpha_,\, d\beta_,} \left(\dfrac{d\phi_,}{d\alpha_,}\right)^{n+1} \left(\dfrac{d\phi_,}{d\beta_,}\right)^{n'+1}\right)}{[n+1]^{n+1}[n'+1]^{n'+1}(z_, + A)^{n+1}(z_, + B)^{n'+1}}; \tag{B$'$}$$

so that we are conducted by this other method to the same expression (O) for the characteristic function of an ordinary optical system, as that which we before obtained by performing the integrations (N). In all these expressions the sign $\Sigma_{(n,n')}{}^{\infty,\,\infty}_{0,\ 0}$ denotes a summation with reference to the variable integers $n$, $n'$, from zero to infinity.

*Case of a Plane System.*

5. A similar analysis may be applied to integrate the partial differential equation

$$\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dz}\right)^2 = \mu^2, \tag{C$'$}$$

to which the equation (A) of this Supplement reduces itself, when we consider a system of rays of ordinary light, contained in the plane of $xz$. In this case, if we put

$$\left.\begin{aligned} x_\prime &= (x - x_{\prime\prime}) \cos . xx_\prime + (z - z_{\prime\prime}) \cos . zx_\prime, \\ z_\prime &= (x - x_{\prime\prime}) \cos . xz_\prime + (z - z_{\prime\prime}) \cos . zz_\prime, \end{aligned}\right\} \tag{D$'$}$$

we may suppose $x_\prime$ $z_\prime$ to be new rectangular coordinates, in the same plane as $x$ $z$, and such that the axis of $z_\prime$ coincides with the direction of some given ray of the system: and we may denote by $\alpha$, $\gamma$, the cosines of the angles which any near ray makes with these new axes, so that

$$\gamma = \sqrt{1 - \alpha^2}.$$

We shall then have for one form of the integral of the partial differential equation (C$'$), the following combination of equations:

$$\mu x_\prime = \frac{dU}{d\alpha}, \quad V + U = \alpha \frac{dU}{d\alpha}, \tag{E$'$}$$

between which $\alpha$ is conceived to be eliminated, and in which

$$\left.\begin{aligned} U = W - \mu\gamma z_\prime = \mu W^{(0)} - \mu z_\prime + \frac{\mu(z_\prime + A)\alpha^2}{2} - \mu\phi; \\ -\phi = \Sigma_{(i)}{}^{\infty}_{0} \,(W^{(i+3)} + z_\prime \gamma^{(2i+4)}); \\ W^{(i+3)} = \alpha^{i+3} \,.\, w_{i+3}; \quad \gamma^{(2i+4)} = \frac{1\,.\,3\,.\,5\,\ldots\,(2i+1)}{2\,.\,4\,.\,6\,\ldots\,(2i+2)} \cdot \frac{\alpha^{2i+4}}{2i+4}; \end{aligned}\right\} \tag{F$'$}$$

$W^{(0)}$, $w_{i+3}$, being constant coefficients in the development of the function $W$, according to the powers of $\alpha$, and $A$ being another constant in that development, namely, the distance upon the given ray, from the point where it touches the caustic curve of the plane system, to the origin of $x_\prime$ and $z_\prime$. The first of the two equations (E$'$) becomes

$$\alpha = \alpha_\prime + \frac{1}{z_\prime + A} \frac{d\phi}{d\alpha},$$

when we put

$$\alpha_{\prime} = \frac{x_{\prime}}{z_{\prime} + A};$$

and gives therefore, by the well-known theorem of Lagrange, for functions of a single variable,

$$F(\alpha) = F_{\prime} + \Sigma_{(n)}\,{}_0^{\infty}\, \frac{\dfrac{d^n}{d\alpha_{\prime}^n} \cdot \dfrac{dF_{\prime}}{d\alpha_{\prime}} \left(\dfrac{d\phi_{\prime}}{d\alpha_{\prime}}\right)^{n+1}}{[n+1]^{n+1}(z_{\prime} + A)^{n+1}}, \tag{G$'$}$$

$F(\alpha)$ denoting any function of $\alpha$, which admits of being developed according to positive integer powers of $\alpha_{\prime}$, and $F_{\prime}$, $\phi_{\prime}$, being formed from $F$, $\phi$, by changing $\alpha$ to $\alpha_{\prime}$. The cosines $\alpha$, $\gamma$, may therefore be thus developed,

$$\left.\begin{aligned} \alpha &= \alpha_{\prime} + \Sigma_{(n)}\,{}_0^{\infty}\, \frac{\dfrac{d^n}{d\alpha_{\prime}^n} \cdot \left(\dfrac{d\phi_{\prime}}{d\alpha_{\prime}}\right)^{n+1}}{[n+1]^{n+1}(z_{\prime} + A)^{n+1}} \\ \gamma &= \gamma_{\prime} + \Sigma_{(n)}\,{}_0^{\infty}\, \frac{\dfrac{d^n}{d\alpha_{\prime}^n} \cdot \dfrac{d\gamma_{\prime}}{d\alpha_{\prime}} \left(\dfrac{d\phi_{\prime}}{d\alpha_{\prime}}\right)^{n+1}}{[n+1]^{n+1}(z_{\prime} + A)^{n+1}} \end{aligned}\right\} \tag{H$'$}$$

if we put

$$\gamma_{\prime} = \sqrt{1 - \alpha_{\prime}^2}.$$

And since $V$ may be thus expressed,

$$V = \mu z_{\prime} - \mu W^{(0)} + \mu(z_{\prime} + A) \int_0^{\alpha_{\prime}} \alpha \, d\alpha_{\prime}, \tag{I$'$}$$

because

$$dV = \mu(\alpha\, dx_{\prime} + \gamma\, dz_{\prime}) = \mu\alpha(z_{\prime} + A)\, d\alpha_{\prime} + \mu(\alpha\alpha_{\prime} + \gamma)\, dz_{\prime},$$

and because $V$ becomes $\mu z_{\prime} - \mu W^{(0)}$ when $\alpha_{\prime} = 0$, we find, finally,

$$\frac{V}{\mu} = z_{\prime} - W^{(0)} + \frac{(z_{\prime} + A)\alpha_{\prime}^2}{2} + \phi_{\prime} + \Sigma_{(n)}\,{}_0^{\infty}\, \frac{\dfrac{d^n}{d\alpha_{\prime}^n} \cdot \left(\dfrac{d\phi_{\prime}}{d\alpha_{\prime}}\right)^{n+2}}{[n+2]^{n+2}(z_{\prime} + A)^{n+1}}. \tag{K$'$}$$

This form (K$'$) for the integral of the partial differential equation (C$'$), may be verified by observing that it satisfies the condition

$$\frac{1}{\mu}\frac{dV}{dz_{\prime}} = \alpha\alpha_{\prime} + \gamma, \tag{L$'$}$$

$V$ being differentiated for $z_{\prime}$ and $\alpha_{\prime}$ as two independent variables; because

$$\frac{d\phi_{\prime}}{dz_{\prime}} = -\,\Sigma_{(i)}\,{}_0^{\infty} \gamma_{\prime}^{(2i+4)} = \gamma_{\prime} - 1 - \frac{\alpha_{\prime}^2}{2}, \quad \frac{d^2\phi_{\prime}}{d\alpha_{\prime}\, dz_{\prime}} = \frac{d\gamma_{\prime}}{d\alpha_{\prime}} + \alpha_{\prime},$$

$$\frac{d^{n+1}}{d\alpha_{\prime}^{n}\, dz_{\prime}} \cdot \left(\frac{d\phi_{\prime}}{d\alpha_{\prime}}\right)^{n+2} = (n+2)\frac{d^n}{d\alpha_{\prime}^{n}} \left\{ \left(\frac{d\phi_{\prime}}{d\alpha_{\prime}}\right)^{n+1} \left(\frac{d\gamma_{\prime}}{d\alpha_{\prime}} + \alpha_{\prime}\right) \right\},$$

$$\frac{d^{n+1}}{d\alpha_{\prime}^{n+1}} \cdot \alpha_{\prime} \left(\frac{d\phi_{\prime}}{d\alpha_{\prime}}\right)^{n+2} = \alpha_{\prime} \frac{d^{n+1}}{d\alpha_{\prime}^{n+1}} \cdot \left(\frac{d\phi_{\prime}}{d\alpha_{\prime}}\right)^{n+2} + (n+1)\frac{d^n}{d\alpha_{\prime}^{n}} \cdot \left(\frac{d\phi_{\prime}}{d\alpha_{\prime}}\right)^{n+2}, \tag{M$'$}$$

and therefore, differentiating (K') as if $\alpha_{\prime}$ were constant,

$$\frac{1}{\mu}\frac{dV}{dz_{\prime}} = \gamma_{\prime} + \alpha_{\prime}^2 + \alpha_{\prime}\, \Sigma_{(n)}\, {}_0^\infty \frac{\dfrac{d^n}{d\alpha_{\prime}^{n}} \cdot \left(\dfrac{d\phi_{\prime}}{d\alpha_{\prime}}\right)^{n+1}}{[n+1]^{n+1}(z_{\prime}+A)^{n+1}} + \Sigma_{(n)}\, {}_0^\infty \frac{\dfrac{d^n}{d\alpha_{\prime}^{n}} \cdot \dfrac{d\gamma_{\prime}}{d\alpha_{\prime}} \left(\dfrac{d\phi_{\prime}}{d\alpha_{\prime}}\right)^{n+1}}{[n+1]^{n+1}(z_{\prime}+A)^{n+1}}, \tag{N$'$}$$

that is, by (H$'$)

$$\frac{1}{\mu}\frac{dV}{dz_{\prime}} = \alpha\alpha_{\prime} + \gamma.$$

*Case of a System of Revolution.*

6. Another particular case of the partial differential equation (A) deserves to be considered specially; namely the case of systems of revolution, symmetric about some single ray. In this case, if we take for the axis of $z$, the ray which is the axis of the system, $V$ will be a function of $z$ and of $x^2 + y^2$; and if we put

$$x^2 + y^2 = \eta, \tag{O$'$}$$

we may in general suppose $V$ developed according to positive integer powers of $\eta$, in a series satisfying the condition,

$$\left(\frac{dV}{dz}\right)^2 + 4\eta\left(\frac{dV}{d\eta}\right)^2 = \mu^2. \tag{P$'$}$$

To integrate this partial differential equation (P$'$), which is a particular case of (A), we may employ the corresponding case of the general system of equations (D), (E), putting for abridgment

$$\alpha^2 + \beta^2 = \epsilon, \tag{Q$'$}$$

and considering the quantities $W$, $U$, as functions of $\epsilon$, which we shall suppose capable of being developed according to positive integer powers of that variable. In this manner we shall obtain

$$\frac{dU}{d\alpha} = 2\alpha\frac{dU}{d\epsilon}, \quad \frac{dU}{d\beta} = 2\beta\frac{dU}{d\epsilon}, \tag{R$'$}$$

and therefore by (D),

$$\mu^2\eta = 4\epsilon\left(\frac{dU}{d\epsilon}\right)^2, \quad V + U = 2\epsilon\frac{dU}{d\epsilon}. \tag{S$'$}$$

We have also by (E),

$$U = W - \mu z\sqrt{1-\epsilon}, \quad \frac{dU}{d\epsilon} = \frac{dW}{d\epsilon} + \frac{\mu z}{2\gamma}, \tag{T$'$}$$

in which

$$\gamma = \sqrt{1-\epsilon};$$

and we may put

$$\left.\begin{aligned} W &= \mu W^{(0)} + \frac{\mu A\epsilon}{2} + \mu\,\Sigma_{(i)}\,{}_0^\infty\,\epsilon^{i+2} w_{2i+4}; \\ \frac{dW}{d\epsilon} &= \frac{\mu A}{2} + \mu\,\Sigma_{(i)}\,{}_0^\infty\,(i+2)\epsilon^{i+1} w_{2i+4}; \\ \frac{dU}{d\epsilon} &= \frac{\mu(z+A)}{2} - \mu\psi; \quad -\psi = \Sigma_{(i)}\,{}_0^\infty U_{i+1}\epsilon^{i+1}; \\ U_{i+1} &= (i+2)w_{2i+4} + \frac{1\,.\,3\,.\,5\,\ldots\,(2i+1)}{2\,.\,4\,.\,6\,\ldots\,(2i+2)}\frac{z}{2}; \end{aligned}\right\} \qquad \text{(U}'\text{)}$$

in which $W^{(0)}$, $w_{2i+4}$, $A$, are constants of the same kind as before, $A$ denoting the distance of the origin from the focus of central rays. Hence, if we put for abridgment,

$$\epsilon_{\prime} = \frac{\eta}{(z+A)^2} = \frac{x^2+y^2}{(z+A)^2}, \qquad \text{(V}'\text{)}$$

$\epsilon_{\prime}$ is an approximate value of $\epsilon$, and we have the following relation between $\epsilon$ and $\epsilon_{\prime}$,

$$\epsilon = \epsilon_{\prime} + \frac{4\epsilon\psi}{z+A} - \frac{4\epsilon\psi^2}{(z+A)^2}, \qquad \text{(W}'\text{)}$$

which gives, by the theorems before referred to,

$$f(\epsilon) = f_{\prime} + \Sigma_{(n)}\,{}_0^\infty\,\frac{4^{n+1}\dfrac{d^n}{d\epsilon_{\prime}^n}\left\{\dfrac{df_{\prime}}{d\epsilon_{\prime}}\left(\epsilon_{\prime}\psi_{\prime} - \dfrac{\epsilon_{\prime}\psi_{\prime}^2}{z+A}\right)^{n+1}\right\}}{[n+1]^{n+1}(z+A)^{n+1}}, \qquad \text{(X}'\text{)}$$

$f(\epsilon)$ being a function of $\epsilon$ and $f_{\prime}$, $\psi_{\prime}$, being formed from $f$, $\psi$, by changing $\epsilon$ to $\epsilon_{\prime}$. We have also, by (S′) (T′) (U′),

$$\begin{aligned} \frac{V}{\mu} &= z - W^{(0)} + \frac{1}{\mu}\int_0^\epsilon \left(\frac{dU}{d\epsilon} + 2\epsilon\frac{d^2U}{d\epsilon^2}\right) d\epsilon \\ &= z - W^{(0)} + \frac{(z+A)\epsilon}{2} - 2\epsilon\psi + \int_0^\epsilon \psi\, d\epsilon, \end{aligned} \qquad \text{(Y}'\text{)}$$

and therefore, by (X′),

$$\begin{aligned} \frac{V}{\mu} &= z - W^{(0)} + \frac{(z+A)\epsilon_{\prime}}{2} - 2\epsilon_{\prime}\psi_{\prime} + \int_0^{\epsilon_{\prime}} \psi_{\prime}\, d\epsilon_{\prime} \\ &\quad + \Sigma_{(n)}\,{}_0^\infty\,\frac{4^{n+1}}{[n+1]^{n+1}(z+A)^{n+1}}\frac{d^n}{d\epsilon_{\prime}^n}\left\{\left(\frac{z+A}{2} - 2\epsilon_{\prime}\frac{d\psi_{\prime}}{d\epsilon_{\prime}} - \psi_{\prime}\right)\left(\epsilon_{\prime}\psi_{\prime} - \frac{\epsilon_{\prime}\psi_{\prime}^2}{z+A}\right)^{n+1}\right\}; \end{aligned} \qquad \text{(Z}'\text{)}$$

in which,

$$\psi_{\prime} = -\,\Sigma_{(i)}\,{}_0^\infty\, U_{i+1}\epsilon_{\prime}^{i+1}; \qquad \frac{d\psi_{\prime}}{d\epsilon_{\prime}} = -\,\Sigma_{(i)}\,{}_0^\infty\,(i+1)U_{i+1}\epsilon_{\prime}^{i};$$

$$\int_0^{\epsilon_{\prime}} \psi_{\prime}\, d\epsilon_{\prime} = -\,\Sigma_{(i)}\,{}_0^\infty\, U_{i+1}\cdot\frac{\epsilon_{\prime}^{i+2}}{i+2}; \quad \epsilon_{\prime} = \frac{\eta}{(z+A)^2}.$$

The development (Z$'$) is one form of the integral of the partial differential equation (P$'$); another form of the same integral may be obtained from the expression (K$'$) for the characteristic function of a plane system, by changing $\alpha_{\prime}$ to $\dfrac{\sqrt{\eta}}{z+A}$, and $z_{\prime}$ to $z$, and supposing $w_{2i+3} = 0$, and is,

$$\frac{V}{\mu} = z - W^{(0)} + \frac{\eta}{2(z+A)} + \phi_{\prime} + 4\,\Sigma_{(n)}\,{}_0^\infty\,\frac{2^n(z+A)^{n+1}}{[n+2]^{n+2}}\,\frac{d^n}{(d\sqrt{\eta})^n}\left(\eta^{1+\frac{n}{2}}\left(\frac{d\phi_{\prime}}{d\eta}\right)^{n+2}\right), \quad \text{(A}''\text{)}$$

in which

$$\phi_{\prime} = -\,\Sigma_{(i)}\,{}_0^\infty\,\frac{\eta^{i+2}}{(z+A)^{2i+4}}\left(w_{2i+4} + \frac{1\,.\,3\,.\,5\,\ldots\,(2i+1)}{2\,.\,4\,.\,6\,\ldots\,(2i+2)}\,\frac{z}{2i+4}\right). \qquad \text{(B}''\text{)}$$

Each of these forms gives, when we neglect $\eta^4$, the following approximate expression for the characteristic function $V$ of a system of ordinary rays, symmetric about the axis of $z$,

$$\frac{V}{\mu} = z - W^{(0)} + \frac{\eta}{2(z+A)} - \frac{(z+8w_4)\eta^2}{8(z+A)^4} - \frac{(z+16w_6)\eta^3}{16(z+A)^6} + \frac{(z+8w_4)^2\eta^3}{8(z+A)^7}; \qquad \text{(C}''\text{)}$$

in which $\eta = x^2+y^2$, and $W^{(0)}$, $A$, $w_4$, $w_6$, are constants in the development of the connected function $W$, such that when we neglect the eighth power of the sine of the angle contained between a near ray, and the axis of revolution of the system, we have

$$\frac{W}{\mu} = W^{(0)} + \frac{A(\alpha^2+\beta^2)}{2} + w_4(\alpha^2+\beta^2)^2 + w_6(\alpha^2+\beta^2)^3, \qquad \text{(D}''\text{)}$$

$\alpha$, $\beta$, being, as before, the cosines of the angles that the near ray makes with the axes of $x$ and $y$, to which it is nearly perpendicular.

*Verification of the Approximate Integral for Systems of Revolution.*

7. The approximate expression (C$''$) for the characteristic function of an optical system of revolution, admits of extensive applications: it is therefore useful to consider other methods, by which it may be obtained or verified. An immediate verification may be derived from the partial differential equation (P$'$) of which (C$''$) ought to be an approximate integral; namely, by computing from (C$''$) the approximate expressions of

$$\frac{1}{\mu^2}\left(\frac{dV}{dz}\right)^2 \quad \text{and} \quad \frac{4\eta}{\mu^2}\left(\frac{dV}{d\eta}\right)^2,$$

and trying whether their sum is unity, when $\eta^4$ is neglected. Putting for this purpose the expression (C″) under the form

$$\frac{V}{\mu} = z - W^{(0)} + \frac{\eta}{2(z+A)} - \frac{\eta^2}{8(z+A)^3} + \frac{\eta^3}{16(z+A)^5} + \frac{(A-8w_4)\eta^2}{8(z+A)^4} + \frac{(32w_4-16w_6-3A)\eta^3}{16(z+A)^6} + \frac{(A-8w_4)^2\eta^3}{8(z+A)^7}, \tag{E$''$}$$

we find, by differentiation,

$$\left.\begin{aligned} \frac{1}{\mu}\frac{dV}{dz} &= 1 - \frac{\eta}{2(z+A)^2} + \frac{3\eta^2}{8(z+A)^4} - \frac{5\eta^3}{16(z+A)^6} - \frac{(A-8w_4)\eta^2}{2(z+A)^5} \\ &\quad - \frac{3(32w_4-16w_6-3A)\eta^3}{8(z+A)^7} - \frac{7(A-8w_4)^2\eta^3}{8(z+A)^8}; \\ \frac{2}{\mu}\frac{dV}{d\eta} &= \frac{1}{z+A} - \frac{\eta}{2(z+A)^3} + \frac{3\eta^2}{8(z+A)^5} + \frac{(A-8w_4)\eta}{2(z+A)^4} \\ &\quad + \frac{3(32w_4-16w_6-3A)\eta^2}{8(z+A)^6} + \frac{3(A-8w_4)^2\eta^2}{4(z+A)^7}; \end{aligned}\right\} \tag{F$''$}$$

and therefore, neglecting $\eta^4$,

$$\left.\begin{aligned} \frac{1}{\mu^2}\left(\frac{dV}{dz}\right)^2 &= 1 - \frac{\eta}{(z+A)^2} + \frac{\eta^2}{(z+A)^4} - \frac{\eta^3}{(z+A)^6} - \frac{(A-8w_4)\eta^2}{(z+A)^5} \\ &\quad + \frac{(11A-112w_4+48w_6)\eta^3}{4(z+A)^7} - \frac{7(A-8w_4)^2\eta^3}{4(z+A)^8}; \\ \frac{4\eta}{\mu^2}\left(\frac{dV}{d\eta}\right)^2 &= \frac{\eta}{(z+A)^2} - \frac{\eta^2}{(z+A)^4} + \frac{\eta^3}{(z+A)^6} + \frac{(A-8w_4)\eta^2}{(z+A)^5} \\ &\quad - \frac{(11A-112w_4+48w_6)\eta^3}{4(z+A)^7} + \frac{7(A-8w_4)^2\eta^3}{4(z+A)^8}; \end{aligned}\right\} \tag{G$''$}$$

expressions of which the sum is unity, as it ought to be. We may remark that the former of these two expressions represents the square of the cosine, and the latter the square of the sine, of the angle which a near ray makes with the axis of revolution of the system.

*Other Method of obtaining the Approximate Integral.*

8. Again, the approximate integral (C″) of the partial differential equation (P′), may be obtained in the following manner. Since $V$ is supposed capable of being developed according to positive integer powers of $\eta$, let us assume

$$\frac{V}{\mu} = V^{(0)} + V^{(1)}\eta + V^{(2)}\eta^2 + V^{(3)}\eta^3, \tag{H$''$}$$

neglecting $\eta^4$, and considering $V^{(0)}$, $V^{(1)}$, $V^{(2)}$, $V^{(3)}$, as functions of $z$, of which the forms are to be determined. To determine these forms, we have, when $\eta = 0$,

$$\left.\begin{array}{llll} V = \mu V^{(0)} & \dfrac{dV}{d\eta} = \mu V^{(1)}; & \dfrac{d^2V}{d\eta^2} = 2\mu V^{(2)}; & \dfrac{d^3V}{d\eta^3} = 6\mu V^{(3)}; \\ \dfrac{dV}{dz} = \mu \dfrac{dV^{(0)}}{dz} & \dfrac{d^2V}{d\eta\, dz} = \mu \dfrac{dV^{(1)}}{dz}; & \dfrac{d^3V}{d\eta^2\, dz} = 2\mu \dfrac{dV^{(2)}}{dz}; & \dfrac{d^4V}{d\eta^3\, dz} = 6\mu \dfrac{dV^{(3)}}{dz}. \end{array}\right\} \tag{I$''$}$$

The equation (P$'$) shews that $\dfrac{dV}{dz} = \pm\mu$, when $\eta = 0$; and $\dfrac{dV}{dz}$ is positive, if we suppose the motion of the light directed from the negative towards the positive part of the axis of $z$; we have therefore, by (I$''$),

$$\frac{dV^{(0)}}{dz} = 1. \tag{K$''$}$$

The equation (P$'$) gives also, by differentiating it with respect to $\eta$,

$$\left.\begin{array}{l} 0 = \dfrac{dV}{dz}\dfrac{d^2V}{d\eta\, dz} + 4\eta \dfrac{dV}{d\eta}\dfrac{d^2V}{d\eta^2} + 2\left(\dfrac{dV}{d\eta}\right)^2; \\ 0 = \dfrac{dV}{dz}\dfrac{d^3V}{d\eta^2\, dz} + \left(\dfrac{d^2V}{d\eta\, dz}\right)^2 + 4\eta \dfrac{dV}{d\eta}\dfrac{d^3V}{d\eta^3} + 4\eta\left(\dfrac{d^2V}{d\eta^2}\right)^2 + 8\dfrac{dV}{d\eta}\dfrac{d^2V}{d\eta^2}; \\ 0 = \dfrac{dV}{dz}\dfrac{d^4V}{d\eta^3\, dz} + 3\dfrac{d^2V}{d\eta\, dz}\dfrac{d^3V}{d\eta^2\, dz} + 4\eta\dfrac{dV}{d\eta}\dfrac{d^4V}{d\eta^4} + 12\eta\dfrac{d^2V}{d\eta^2}\dfrac{d^3V}{d\eta^3} + 12\dfrac{dV}{d\eta}\dfrac{d^3V}{d\eta^3} + 12\left(\dfrac{d^2V}{d\eta^2}\right)^2; \end{array}\right\} \tag{L$''$}$$

and, making $\eta = 0$, we find by (I$''$) the following equations in ordinary differentials, from which $V^{(1)}$, $V^{(2)}$, $V^{(3)}$, are to be deduced:

$$\left.\begin{array}{l} 0 = \dfrac{dV^{(1)}}{dz} + 2V^{(1)2}; \\ 0 = \dfrac{dV^{(2)}}{dz} + 8V^{(1)}V^{(2)} + \tfrac{1}{2}\left(\dfrac{dV^{(1)}}{dz}\right)^2; \\ 0 = \dfrac{dV^{(3)}}{dz} + 12V^{(1)}V^{(3)} + \dfrac{dV^{(1)}}{dz}\dfrac{dV^{(2)}}{dz} + 8V^{(2)2}. \end{array}\right\} \tag{M$''$}$$

These three differential equations, when divided respectively by $V^{(1)2}$, $V^{(1)4}$, $V^{(1)6}$, can easily be integrated, and give, when combined with the integral of (K$''$),

$$\left.\begin{array}{l} v_0 = V^{(0)} - z; \\ v_1 = \dfrac{1}{V^{(1)}} - 2z; \\ v_2 = \dfrac{V^{(2)}}{V^{(1)4}} + 2z; \\ v_3 = \dfrac{V^{(3)}}{V^{(1)6}} - 4z - \dfrac{4(v_1 + v_2)^2}{2z + v_1}; \end{array}\right\} \tag{N$''$}$$

$v_0$, $v_1$, $v_2$, $v_3$, being the four arbitrary constants introduced by the four integrations. The functions $V^{(0)}$, $V^{(1)}$, $V^{(2)}$, $V^{(3)}$, are therefore of the form

$$\left.\begin{aligned} V^{(0)} &= z + v_0; \quad V^{(1)} = \frac{1}{2z + v_1}; \quad V^{(2)} = \frac{v_2 - 2z}{(2z + v_1)^4}; \\ V^{(3)} &= \frac{4z + v_3}{(2z + v_1)^6} + \frac{4(v_1 + v_2)^2}{(2z + v_1)^7}; \end{aligned}\right\} \qquad (\mathrm{O}'')$$

and these forms for the coefficients of the development (H″), agree perfectly with the development (C″) or (E″), when we establish the following relations between the constants:

$$v_0 = -W^{(0)}; \quad v_1 = 2A; \quad v_2 = -16w_4; \quad v_3 = 8(16w_4 - 8w_6 - A): \qquad (\mathrm{P}'')$$

we see, therefore, that the present method of integration confirms the former results.

*Connexion of the Longitudinal Aberration, in a System of Revolution, with the Development of the Characteristic Function $V$.*

9. To give now an example of the optical use of the development which has been thus obtained, let us consider its connexion with the aberrations of the near rays, from the principal or central focus. We have already remarked that the constant $A$ denotes the distance of the origin of coordinates, upon the central ray, beyond this principal focus, in such a manner that the focal ordinate is $= -A$. For the ordinate $Z$, of intersection of any near ray with the central ray, we have by the fourth of the equations (C), of the present Supplement,

$$Z = \frac{1}{\mu(\alpha^2 + \beta^2)} \frac{\delta W}{\delta \gamma}, \qquad (\mathrm{Q}'')$$

if we form the coefficient of $\dfrac{\delta W}{\delta \gamma}$ by putting $W$ under the form of a homogeneous function of $\alpha$, $\beta$, $\gamma$, of the dimension zero, with the help of the relation $\alpha^2 + \beta^2 + \gamma^2 = 1$, and then by differentiating this function, as if $\alpha$, $\beta$, were constant, and $\gamma$ the only variable. Employing therefore for $\dfrac{W}{\mu}$ the development

$$\frac{W}{\mu} = W^{(0)} + \frac{A(\alpha^2 + \beta^2)}{2(\alpha^2 + \beta^2 + \gamma^2)} + \Sigma_{(i)}{}_0^\infty \frac{w_{2i+4}(\alpha^2 + \beta^2)^{i+2}}{(\alpha^2 + \beta^2 + \gamma^2)^{i+2}}, \qquad (\mathrm{R}'')$$

which is of the homogeneous form required, and, after differentiating for $\gamma$, making $\alpha^2 + \beta^2 + \gamma^2 = 1$, we find for the ordinate $Z$,

$$Z = -A + A(1 - \gamma) - \gamma\, \Sigma_{(i)}{}_0^\infty\, (2i + 4) w_{2i+4} (\alpha^2 + \beta^2)^{i+1}, \qquad (\mathrm{S}'')$$

a series of which the term $-A$ being the ordinate of the central focus, the remainder is the longitudinal aberration: $\gamma$ is the cosine of the angle which the near ray makes with the central ray, and $\alpha^2+\beta^2$ is the square of the sine of that angle. If therefore we denote the aberration $Z+A$ by $\Lambda$, we may develope $\Lambda$ in a series of the form

$$\Lambda = L(\alpha^2+\beta^2) + L_1(\alpha^2+\beta^2)^2 + \&\text{c.}, \tag{T$''$}$$

in which

$$L = \tfrac{1}{2}A - 4w_4, \quad L_1 = \tfrac{1}{8}A + 2w_4 - 6w_6. \tag{U$''$}$$

And if by these relations (U$''$), we eliminate $w_4$, $w_6$ from the approximate expression (E$''$), we find the following formula:

$$\begin{aligned} V = \mu(z - W^{(0)}) &+ \frac{\mu\eta}{2(z+A)} + \frac{\mu\eta^2}{4(z+A)^3}\left(\frac{L}{z+A} - \tfrac{1}{2}\right) \\ &+ \frac{\mu\eta^3}{12(z+A)^5}\left\{\frac{2L_1-5L}{z+A} + \frac{6L^2}{(z+A)^2} + \tfrac{3}{4}\right\}, \end{aligned} \tag{V$''$}$$

which shews the connexion in a system of revolution between the development of the longitudinal aberration $\Lambda$, and that of the characteristic function $V$.

*Changes of a System of Revolution, produced by Ordinary Refraction.*

10. Suppose now that the rays of this system of revolution fall upon a refracting surface of revolution, having for axis the axis of the system, and having for equation

$$z = z_0 + z_1\eta + z_2\eta^2 + z_3\eta^3 + \&\text{c.}, \tag{W$''$}$$

in which $\eta$ is still $= x^2+y^2 =$ the square of the perpendicular distance of a point $x\ y\ z$, from the axis; and let $\mu'$ be the refracting index of the new medium into which the rays pass after refraction. It is evident that in this new medium, the rays will compose a new system of revolution, symmetric about the same axis as before; and we may in general suppose the characteristic function $V'$ of this new system, which is analogous to $V$ of the old, developed in a series similar to (V$''$),

$$\begin{aligned} V' = \mu'(z - W'^{(0)}) &+ \frac{\mu'\eta}{2(z+A')} + \frac{\mu'\eta^2}{4(z+A')^3}\left(\frac{L'}{z+A'} - \tfrac{1}{2}\right) \\ &+ \frac{\mu'\eta^3}{12(z+A')^5}\left\{\frac{2L_1'-5L'}{z+A'} + \frac{6L'^2}{(z+A')^2} + \tfrac{3}{4}\right\}: \end{aligned} \tag{X$''$}$$

the constants $A'\ L'\ L_1'$ being similar to $A\ L\ L_1$, in such a manner that the ordinate $Z'$ of intersection of the axis with a near ray, is

$$Z' = -A' + L'(\alpha'^2+\beta'^2) + L_1'(\alpha'^2+\beta'^2)^2, \tag{Y$''$}$$

if $\alpha'^2+\beta'^2$ denote the square of the sine of the angle which the near ray makes with the axis, and if we neglect the sixth power of this sine. To connect the new and old constants in the development of the characteristic function, we have, by the nature of this function, and by the principles of my former memoirs, the condition

$$0 = \Delta V = V' - V; \tag{Z$''$}$$

which is to be satisfied for all the points of the refracting surface, and which may therefore be differentiated, considering $\Delta V$ as a function of $z$, $\eta$, namely the difference of the developments (V$''$) (X$''$), and considering $z$ as itself a function of $\eta$, assigned by the equation of the refracting surface (W$''$). In this manner we find, transposing the symbols $\Delta$, $d$,

$$\left.\begin{aligned}
&0 = \Delta\frac{dV}{d\eta} + \frac{dz}{d\eta}\,\Delta\frac{dV}{dz};\\
&0 = \Delta\frac{d^2V}{d\eta^2} + 2\frac{dz}{d\eta}\,\Delta\frac{d^2V}{dz\,d\eta} + \left(\frac{dz}{d\eta}\right)^2\Delta\frac{d^2V}{dz^2} + \frac{d^2z}{d\eta^2}\,\Delta\frac{dV}{dz};\\
&0 = \Delta\frac{d^3V}{d\eta^3} + 3\frac{dz}{d\eta}\,\Delta\frac{d^3V}{dz\,d\eta^2} + 3\left(\frac{dz}{d\eta}\right)^2\Delta\frac{d^3V}{dz^2\,d\eta} + \left(\frac{dz}{d\eta}\right)^3\Delta\frac{d^3V}{dz^3}\\
&\qquad + 3\frac{d^2z}{d\eta^2}\,\Delta\frac{d^2V}{dz\,d\eta} + 3\frac{dz}{d\eta}\frac{d^2z}{d\eta^2}\,\Delta\frac{d^2V}{dz^2} + \frac{d^3z}{d\eta^3}\,\Delta\frac{dV}{dz};
\end{aligned}\right\} \tag{A$'''$}$$

and, making after the differentiations $\eta = 0$, we have

$$\left.\begin{gathered}
z = z_0; \quad \frac{dz}{d\eta} = z_1; \quad \frac{d^2z}{d\eta^2} = 2z_2; \quad \frac{d^3z}{d\eta^3} = 6z_3;\\
\Delta\frac{dV}{dz} = \Delta\mu; \quad \Delta\frac{d^2V}{dz^2} = 0; \quad \Delta\frac{d^3V}{dz^3} = 0;\\
\Delta\frac{dV}{d\eta} = \tfrac{1}{2}\Delta\,.\,\frac{\mu}{z+A}; \quad \Delta\frac{d^2V}{dz\,d\eta} = \tfrac{1}{2}\Delta\,.\,\frac{-\mu}{(z+A)^2}; \quad \Delta\frac{d^3V}{dz^2\,d\eta} = \Delta\,.\,\frac{\mu}{(z+A)^3};\\
\Delta\frac{d^2V}{d\eta^2} = \Delta\left\{\frac{\mu L}{2(z+A)^4} - \frac{\mu}{4(z+A)^3}\right\}; \quad \Delta\frac{d^3V}{dz\,d\eta^2} = \Delta\left\{\frac{-2\mu L}{(z+A)^5} + \frac{3\mu}{4(z+A)^4}\right\};\\
\Delta\frac{d^3V}{d\eta^3} = \Delta\,.\,\frac{\mu}{(z+A)^5}\left(\frac{2L_1 - 5L}{2(z+A)} + \frac{3L^2}{(z+A)^2} + \tfrac{3}{8}\right).
\end{gathered}\right\} \tag{B$'''$}$$

We have therefore, I[st], for the change of $-A$, the ordinate of the central focus,

$$0 = \Delta\frac{\mu}{z+A} + 2z_1\,\Delta\mu: \tag{C$'''$}$$

II[nd], for the change of $L$, the first or principal coefficient of aberration,

$$0 = \Delta\left\{\frac{2\mu L}{(z+A)^4} - \frac{\mu}{(z+A)^3}\right\} - 4z_1\,\Delta\,.\,\frac{\mu}{(z+A)^2} + 8z_2\,\Delta\mu: \tag{D$'''$}$$

III[rd], for the change of $L_1$, the coefficient of the fourth power of the sine of the angular aberration, in the expression of the longitudinal,

$$\begin{aligned}
0 = \Delta\,.\,\frac{\mu}{(z+A)^5}\left(\frac{2L_1 - 5L}{2(z+A)} + \frac{3L^2}{(z+A)^2} + \tfrac{3}{8}\right) + 3z_1\,\Delta\,.\,\frac{\mu}{(z+A)^4}\left(\frac{-2L}{z+A} + \tfrac{3}{4}\right)\\
+ 3z_1^2\,\Delta\,.\,\frac{\mu}{(z+A)^3} - 3z_2\,\Delta\,.\,\frac{\mu}{(z+A)^2} + 6z_3\,\Delta\mu;
\end{aligned} \tag{E$'''$}$$

$z$ being the ordinate of the point of central incidence. With respect to the present meaning of the sign $\Delta$, we may remark, that the first of the three equations (C‴) (D‴) (E‴) is equivalent to the following:

$$\frac{\mu}{z+A} + 2z_1\mu = \frac{\mu'}{z+A'} + 2z_1\mu'; \tag{F‴}$$

and the two others are to be similarly interpreted.

*Example; Spheric Refraction; Mr. Herschel's Formula for the Aberration of a thin Lens.*

11. These general equations for refracting surfaces of revolution may be adapted to the case of a refracting spheric surface, by making

$$z_1 = \frac{1}{2r}; \quad z_2 = \frac{1}{8r^3}; \quad z_3 = \frac{1}{16r^5}; \tag{G‴}$$

the two first, for example, becoming

$$0 = \Delta\left(\frac{\mu}{z+A} + \frac{\mu}{r}\right); \tag{H‴}$$

$$0 = \Delta\left(\frac{2\mu L}{(z+A)^4} - \frac{\mu}{(z+A)^3} - \frac{2\mu}{r(z+A)^2} + \frac{\mu}{r^3}\right); \tag{I‴}$$

which contain under a convenient form, the known theorems for the change of a central focus, and of the principal coefficient of aberration, by refraction of a spheric surface; $r$ being the radius of this surface, and being considered as positive or negative, according as the convexity or concavity is turned towards the incident rays.

If, for instance, we consider an infinitely thin lens in vacuo, having $\mu$ for its refractive index, and having $r$, $r'$, for the radii of its two spheric surfaces, (positive when those surfaces are convex towards the incident rays,) we may take the point of central incidence for origin, and the equation (H‴) will become,

$$0 = \frac{\mu}{A'} - \frac{1}{A} + \frac{\mu-1}{r}; \quad 0 = \frac{1}{A''} - \frac{\mu}{A'} + \frac{1-\mu}{r'}, \tag{K‴}$$

$-A$, $-A'$, $-A''$, being the ordinates of the central focus in the three successive states of the system; and similarly, (I‴) will give

$$\left.\begin{aligned} 0 &= \frac{2\mu L'}{A'^4} - \frac{2L}{A^4} - \left(\frac{\mu}{A'^3} - \frac{1}{A^3}\right) - \left(\frac{2\mu}{rA'^2} - \frac{2}{rA^2}\right) + \frac{\mu-1}{r^3};\\ 0 &= \frac{2L''}{A''^4} - \frac{2\mu L'}{A'^4} - \left(\frac{1}{A''^3} - \frac{\mu}{A'^3}\right) - \left(\frac{2}{r'A''^2} - \frac{2\mu}{r'A'^2}\right) + \frac{1-\mu}{r'^3}; \end{aligned}\right\} \tag{L‴}$$

$L$, $L'$, $L''$, being the three successive values of the principal coefficient of aberration. Adding the two equations (L‴), the intermediate coefficient $L'$ disappears, and we find,

$$0 = \frac{2L''}{A''^4} - \frac{2L}{A^4} - \left(\frac{1}{A''^3} - \frac{1}{A^3}\right) + \frac{2\mu}{A'^2}\left(\frac{1}{r'} - \frac{1}{r}\right) - \left(\frac{2}{r'A''^2} - \frac{2}{rA^2}\right) + (\mu-1)\left(\frac{1}{r^3} - \frac{1}{r'^3}\right), \tag{M‴}$$

in which, by (K‴),

$$\frac{1}{A'} = \frac{1}{\mu A} - \frac{\mu - 1}{\mu r}; \quad \frac{1}{A''} = \frac{1}{A} + (\mu - 1)\left(\frac{1}{r'} - \frac{1}{r}\right); \tag{N‴}$$

and therefore,

$$\frac{L''}{A''^4} - \frac{L}{A^4} = \left(\frac{\mu - 1}{2}\right)\left(\frac{1}{r'} - \frac{1}{r}\right)\left(M^{(0)} + \frac{M^{(1)}}{A} + \frac{M^{(2)}}{A^2}\right), \tag{O‴}$$

if we put for abridgment

$$\left.\begin{aligned} M^{(0)} &= \frac{\mu^2}{r'^2} + \frac{1 + 2\mu - 2\mu^2}{r'r} + \frac{\dfrac{2}{\mu} - 2\mu + \mu^2}{r^2}; \\ M^{(1)} &= \frac{1 + 3\mu}{r'} + \frac{\dfrac{4}{\mu} + 3 - 3\mu}{r}; \\ M^{(2)} &= \frac{2}{\mu} + 3. \end{aligned}\right\} \tag{P‴}$$

It is easy to see that the formula (O‴) agrees with the expression for the spherical aberration of an infinitely thin lens, which Mr. Herschel has deduced by reasonings of a different kind, in his memoir "On Aberrations of Compound Lenses and Object-Glasses," published in the second part of the *Philosophical Transactions* for the year 1821; and in his excellent "Treatise on Light," published in the *Encyclopædia Metropolitana.*

The elegance of this formula of Mr. Herschel, and the important consequences which he has obtained from it, have induced us to shew how the same expression may be derived from the development of the characteristic function of an ordinary system of revolution, assigned in the present Supplement. The same form of development, and those other forms which we have assigned in the same Supplement, for systems not of revolution, contain the solution of other optical problems, of which we hope to treat hereafter.