SUPPLEMENT TO AN ESSAY ON THE THEORY OF SYSTEMS OF RAYS

By

William Rowan Hamilton

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NOTE ON THE TEXT

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In the original publication, the word *arêtes* was spelt as *aretes* in article 8, and as *arétes* in article 9.

David R. Wilkins

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Supplement to an Essay on the Theory of Systems of Rays. By WILLIAM R. HAMILTON, A. B., M. R. I. A., *M. Ast. Soc. Lond., Hon. M. Soc. Arts for Scotland, Andrews' Professor of Astronomy in the University of Dublin, and Royal Astronomer of Ireland.*

Read April 26, 1830.

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INTRODUCTION.

The present Supplement contains some developments of a view of Mathematical Optics, which was proposed by me in the foregoing volume of the Transactions of this Academy. According to that view, the geometrical properties of an optical system of rays, whether straight or curved, whether ordinary or extraordinary, may be deduced by analytic methods, from one fundamental formula, and one characteristic function: the formula being an expression for the variation which the definite integral, called *action*, receives, when the coordinates of its limits vary; and the characteristic function being this integral itself, considered as depending on those coordinates. Although this view was stated, and the formula announced, in the Table of Contents prefixed to my preceding Memoir, yet the demonstration was not given in the part already published, except for the Systems produced by the ordinary reflection of light; it has therefore been thought advisable to give in the present Supplement, the general demonstration of the formula, and some of its general consequences. The demonstration is founded on the principles of the calculus of variations, and on the known optical principle of least action. The result deduced from these principles, is, that the coefficients of the variations of the final coordinates, in the variation of the integral called action, are equal to the coefficients of the variations of the cosines of the angles which the element of the ray makes with the axes of coordinates, in the variation of a certain homogeneous function of those cosines: this homogeneous function, which is of the first dimension, being equal to the multiplier of the element of the ray under the integral sign, and therefore to the velocity of that element, estimated on the hypothesis of emission. It was proposed, in my former Memoir, to call this result the *principle of constant action*: partly to mark its connexion with the known law of *least action*, and partly because it gives immediately the differential equation of that important class of surfaces, which, on the hypothesis of undulation are called *waves*, and which, on the hypothesis of molecular emission may be named *surfaces of constant action*. But in the present Supplement, it is proposed to designate the fundamental formula by the less hypothetical name of the *Equation of the Characteristic Function*: because, whatever may be the nature of light, the definite integral in this equation is, as we have before observed, a function of the coordinates of its limits, on the analytic form of which function the properties of the system depend. In the applications of this formula, to systems of straight rays, ordinary or extraordinary, it is advantageous to introduce the consideration of a characteristic function of another kind, depending on the direction rather than on the coordinates of the ray, but connected with the former function, and with the geometrical properties of the system, by relations which form the chief subject of the present Memoir. The theory of these relations, from the generality of its nature, will, perhaps, be interesting to Mathematicians: I am aware that it admits of being much farther extended, and that much remains to be done, in order to render it practically useful.

WILLIAM R. HAMILTON.

OBSERVATORY, *April* 1830.

CONTENTS OF THE SUPPLEMENT.

On osculating spheroids, and surfaces of constant action; the extreme spheroids have their centres upon the caustic surfaces, and their planes of osculation touch the developable pencils; the centre of the osculating spheroid coincides with the focus of the osculating focal reflector or refractor, when the point of contact and the plane of osculation are the same; the principal foci are centres of spheroids which have complete contact of the second order with the surfaces of constant action; they are also foci of osculating focal reflectors or refractors which have contact of the same kind with the last given reflector or refractor;

$$\delta^2 V = \delta^2 V'; \quad \frac{\rho_1^{-1} - \rho^{-1}}{\rho^{-1} - \rho_2^{-1}} = \zeta \left(\frac{\sin.(\psi - \psi_1)}{\sin.(\psi_2 - \psi)} \right)^2; \qquad 15, 16.$$

On foci by projection and virtual foci; the planes of extreme projection coincide with the planes of extreme virtual foci; these planes are perpendicular to each other, and furnish in general a pair of natural coordinates; the tangent planes to the developable pencils are symmetrically situated with respect to these natural coordinate planes, and coincide with them when the system is rectangular; this latter condition is satisfied in ordinary systems, and the osculating foci of such systems coincide with the foci by projection;

$$p = -\frac{\delta x'^2 + \delta y'^2 + \delta z'^2}{\delta\alpha\,\delta x' + \delta\beta\,\delta y' + \delta\gamma\,\delta z'}; \quad \frac{1}{p} = \frac{(\cos.\Pi)^2}{p_1} + \frac{(\sin.\Pi)^2}{p_2};$$

$$r = -\frac{\delta\alpha\,\delta x_{,} + \delta\beta\,\delta y_{,} + \delta\gamma\,\delta z_{,}}{\delta\alpha^2 + \delta\beta^2 + \delta\gamma^2}; \quad r = r_1(\cos.\omega)^2 + r_2(\sin.\omega)^2; \qquad 17, 18.$$

Theory of lateral aberrations; aberration from a principal focus; elliptic or hyperbolic lines of uniform condensation; natural axes of the system;

$$\Delta x = [\delta x] + \frac{1}{2}[\delta^2 x] + \frac{1}{2\,.\,3}[\delta^3 x] + \&\text{c.};$$

$$M'' = \left\{ \frac{\delta^3 W}{\delta\alpha^3}\frac{\delta^3 W}{\delta\alpha\,\delta\beta^2} - \left(\frac{\delta^3 W}{\delta\alpha^2\,\delta\beta}\right)^2 \right\} a^2 + \left\{ \frac{\delta^3 W}{\delta\alpha^2\,\delta\beta}\frac{\delta^3 W}{\delta\beta^3} - \left(\frac{\delta^3 W}{\delta\alpha\,\delta\beta^2}\right)^2 \right\} b^2;$$

$$\frac{\delta^3 W}{\delta\alpha^3}\frac{\delta^3 W}{\delta\beta^3} = \frac{\delta^3 W}{\delta\alpha^2\,\delta\beta}\frac{\delta^3 W}{\delta\alpha\,\delta\beta^2}; \qquad 19, 20.$$

SUPPLEMENT, &c. &c.

Fundamental Formula of Optical Systems, or Equation of the Characteristic Function.

1. The fundamental formula that we shall employ in our investigations respecting the geometrical properties of optical systems of rays, straight or curved, ordinary or extraordinary, which, after issuing from any luminous origin, have been any number of times reflected and refracted by any combination of media, according to any laws compatible with the known condition of least action, is the following:

$$\delta \int v\, ds = \frac{\delta v}{\delta \alpha}\, \delta x + \frac{\delta v}{\delta \beta}\, \delta y + \frac{\delta v}{\delta \gamma}\, \delta z. \tag{A}$$

In this equation, x, y, z, are the coordinates of any point of the system, referred to three rectangular axes; α, β, γ, are the cosines of the angles which the tangent to the ray at that point, or the direction of the element ds, makes with the axes of coordinates; v is the quantity which in the hypothesis of molecular emission represents the velocity of this element, and is supposed to be in general a function of the six quantities x, y, z, α, β, γ, depending on the nature of the medium, and involving also the colour of the light; the partial differential coefficients,

$$\frac{\delta v}{\delta \alpha}, \quad \frac{\delta v}{\delta \beta}, \quad \frac{\delta v}{\delta \gamma},$$

are obtained by putting v under the form of a homogeneous function of α, β, γ, of the first dimension, with the help of the relation $\alpha^2 + \beta^2 + \gamma^2 = 1$, and by then differentiating this homogeneous function, as if α, β, γ, were three independent variables; finally, the definite integral $\int v\, ds$ is taken from the luminous origin to the point x, y, z, and the variation $\delta \int v\, ds$ is obtained by supposing the coordinates of this last point to receive any infinitely small changes, the colour remaining the same.

2. To deduce the equation (A) from the known condition of least action, let us observe that by the calculus of variations,

$$\delta \int v\, ds = \int (\delta v \,.\, ds + v \,.\, \delta ds);$$

in which, by what we have laid down respecting the form of v,

$$\delta v = \frac{\delta v}{\delta x}\, \delta x + \frac{\delta v}{\delta y}\, \delta y + \frac{\delta v}{\delta z}\, \delta z + \frac{\delta v}{\delta \alpha}\, \delta \alpha + \frac{\delta v}{\delta \beta}\, \delta \beta + \frac{\delta v}{\delta \gamma}\, \delta \gamma,$$

$$v = \alpha \frac{\delta v}{\delta \alpha} + \beta \frac{\delta v}{\delta \beta} + \gamma \frac{\delta v}{\delta \gamma};$$

while, by the nature of α, β, γ,

$$\begin{aligned}
\delta\alpha \,.\, ds + \alpha \,.\, \delta ds &= \delta \,.\, \alpha\, ds = \delta \,.\, dx = d \,.\, \delta x,\\
\delta\beta \,.\, ds + \beta \,.\, \delta ds &= \delta \,.\, \beta\, ds = \delta \,.\, dy = d \,.\, \delta y,\\
\delta\gamma \,.\, ds + \gamma \,.\, \delta ds &= \delta \,.\, \gamma\, ds = \delta \,.\, dz = d \,.\, \delta z;
\end{aligned}$$

we have therefore,

$$\begin{aligned}
\delta \int v\, ds &= \int \left(\frac{\delta v}{\delta x}\,\delta x + \frac{\delta v}{\delta y}\,\delta y + \frac{\delta v}{\delta z}\,\delta z \right) ds + \int \left(\frac{\delta v}{\delta \alpha}\, d\delta x + \frac{\delta v}{\delta \beta}\, d\delta y + \frac{\delta v}{\delta \gamma}\, d\delta z \right)\\
&= \frac{\delta v}{\delta \alpha}\,\delta x - \frac{\delta v'}{\delta \alpha'}\,\delta x' + \frac{\delta v}{\delta \beta}\,\delta y - \frac{\delta v'}{\delta \beta'}\,\delta y' + \frac{\delta v}{\delta \gamma}\,\delta z - \frac{\delta v'}{\delta \gamma'}\,\delta z'\\
&\quad + \int \delta x \left(\frac{\delta v}{\delta x}\, ds - d\frac{\delta v}{\delta \alpha} \right) + \int \delta y \left(\frac{\delta v}{\delta y}\, ds - d\frac{\delta v}{\delta \beta} \right) + \int \delta z \left(\frac{\delta v}{\delta z}\, ds - d\frac{\delta v}{\delta \gamma} \right),
\end{aligned}$$

the accented quantities belonging to the first limit of integral, and disappearing when that limit is fixed. The condition of least action requires that the quantities which remain under the integral sign, as coefficients of δx, δy, δz, should also vanish, and furnishes thereby the following general differential equations of a ray,

$$\frac{\delta v}{\delta x}\, ds = d\frac{\delta v}{\delta \alpha}, \quad \frac{\delta v}{\delta y}\, ds = d\frac{\delta v}{\delta \beta}, \quad \frac{\delta v}{\delta z}\, ds = d\frac{\delta v}{\delta \gamma}, \tag{B}$$

of which any two include the third. And rejecting the evanescent quantities in the expression for $\delta \int v\, ds$, we find the formula (A), which it was required to demonstrate.

3. The fundamental formula thus obtained, resolves itself into the three following equations:

$$\frac{\delta \int v\, ds}{\delta x} = \frac{\delta v}{\delta \alpha}, \quad \frac{\delta \int v\, ds}{\delta y} = \frac{\delta v}{\delta \beta}, \quad \frac{\delta \int v\, ds}{\delta z} = \frac{\delta v}{\delta \gamma},$$

which we shall thus write:

$$\frac{\delta V}{\delta x} = \frac{\delta v}{\delta \alpha}, \quad \frac{\delta V}{\delta y} = \frac{\delta v}{\delta \beta}, \quad \frac{\delta V}{\delta z} = \frac{\delta v}{\delta \gamma}, \tag{C}$$

representing, for abridgment, the definite integral $\displaystyle\int v\, ds$ by V, and considering the integral as a function of x, y, z, of which the form depends upon the nature of the system, the medium, and the light, and of which the partial differential coefficients of the first order are denoted by

$$\frac{\delta V}{\delta x}, \quad \frac{\delta V}{\delta y}, \quad \frac{\delta V}{\delta z}.$$

When the form of V is given, we can obtain these coefficients by differentiation; and if we know also the form of v, which depends only on the nature of the medium and of the light, we can by the equations (C) determine α, β, γ, as functions of x, y, z; that is, we can find the direction of the ray or rays passing through any proposed point of the system. The geometrical properties of one system as distinguished from another, for any given medium and any given kind of light, may therefore be deduced by analytic reasonings from the form of the function V; on which account we shall call this function V, the *characteristic function of the system*; and the fundamental formula (A), that expresses its variation, namely:

$$\delta V = \frac{\delta v}{\delta \alpha}\,\delta x + \frac{\delta v}{\delta \beta}\,\delta y + \frac{\delta v}{\delta \gamma}\,\delta z,$$

we shall call the *equation of the characteristic function.*

Other Characteristic Function for Systems of Straight Rays.

4. In the remaining reasonings of the present Supplement, we shall confine ourselves to the consideration of homogeneous systems of straight rays not parallel; and in investigating the properties of such systems, it will be useful to employ another function, connected with the function V by many remarkable relations. This new function, which we shall call W, is determined by the condition:

$$W + V = x\frac{\delta v}{\delta \alpha} + y\frac{\delta v}{\delta \beta} + z\frac{\delta v}{\delta \gamma}, \tag{D}$$

which gives, on account of (A), or (C),

$$\delta W = x\,\delta\frac{\delta v}{\delta \alpha} + y\,\delta\frac{\delta v}{\delta \beta} + z\,\delta\frac{\delta v}{\delta \gamma}. \tag{E}$$

It results from this differential equation (E) (in which we employ the sign of variation δ to mark the connexion with the definite integral $\int v\,ds$, a remark which applies to the whole of the present Supplement,) that if the variations of x, y, z, be such as to leave α, β, γ, and consequently

$$\frac{\delta v}{\delta \alpha}, \quad \frac{\delta v}{\delta \beta}, \quad \frac{\delta v}{\delta \gamma}$$

unchanged, that is, if we pass from any one point of the system to any other point situated upon the same ray, the function W will not vary. We may, therefore, consider W as a function of α, β, γ, of which the form can be determined from that of V, by eliminating x, y, z, between the equations (C) and (D), when the nature of the medium and of the light is known. Reciprocally, if the connexion between W, α, β, γ, be given, that which exists between V, x, y, z, can be found. For if we suppose that for the sake of symmetry, W has been put under the form of a homogeneous function of the dimension i, by the help of the relation $\alpha^2 + \beta^2 + \gamma^2 = 1$, and then differentiated as if α, β, γ, were three independent variables, we shall have, by (E), and by the nature of homogeneous functions,

$$\begin{aligned}
\frac{\delta W}{\delta \alpha} &= iW\alpha + x\frac{\delta^2 v}{\delta \alpha^2} + y\frac{\delta^2 v}{\delta \alpha\,\delta \beta} + z\frac{\delta^2 v}{\delta \alpha\,\delta \gamma},\\
\frac{\delta W}{\delta \beta} &= iW\beta + x\frac{\delta^2 v}{\delta \alpha\,\delta \beta} + y\frac{\delta^2 v}{\delta \beta^2} + z\frac{\delta^2 v}{\delta \beta\,\delta \gamma},\\
\frac{\delta W}{\delta \gamma} &= iW\gamma + x\frac{\delta^2 v}{\delta \alpha\,\delta \gamma} + y\frac{\delta^2 v}{\delta \beta\,\delta \gamma} + z\frac{\delta^2 v}{\delta \gamma^2},
\end{aligned}$$

in which we shall for simplicity suppose the dimension $i = 0$; and eliminating α, β, γ, by means of these equations, from that marked (D), we shall deduce the relation between V, x, y, z, from the relation between W, α, β, γ. We may therefore consider W as itself a characteristic function, which distinguishes any one homogeneous system of straight rays not parallel, from any other such system, composed of light of the same kind, and contained in the same medium. It is evident that on some occasions it must be advantageous to attend to the function W instead of V, because V changes in passing from one point to another of the same ray, whereas W is constant, when the ray and the system are given. On the other hand, in any sudden change of the system by reflection or refraction, the function W receives a sudden alteration, while the change of V is gradual; it is therefore convenient to employ V instead of W, in investigating the effects of such changes. Accordingly, in the remainder of this memoir, we shall consider both these functions, and examine the relations between them: and shall begin by investigating the connexions between their partial differential coefficients.

Connexions between the Partial Differential Coefficients of the two Characteristic Functions.

5. The connexions between these coefficients, are to be obtained by differentiating the preceding expressions for

$$\frac{\delta V}{\delta x}, \quad \frac{\delta V}{\delta y}, \quad \frac{\delta V}{\delta z}, \quad \frac{\delta W}{\delta \alpha}, \quad \frac{\delta W}{\delta \beta}, \quad \frac{\delta W}{\delta \gamma},$$

and by attending to the homogeneous forms which we have assigned to v and W. The dimension of W being supposed $= 0$, we have by the nature of homogeneous functions,

$$\left.\begin{aligned}
\alpha \frac{\delta W}{\delta \alpha} + \beta \frac{\delta W}{\delta \beta} + \gamma \frac{\delta W}{\delta \gamma} &= 0; \\
\frac{\delta W}{\delta \alpha} + \alpha \frac{\delta^2 W}{\delta \alpha^2} + \beta \frac{\delta^2 W}{\delta \alpha\, \delta \beta} + \gamma \frac{\delta^2 W}{\delta \alpha\, \delta \gamma} &= 0; \\
\frac{\delta W}{\delta \beta} + \alpha \frac{\delta^2 W}{\delta \alpha\, \delta \beta} + \beta \frac{\delta^2 W}{\delta \beta^2} + \gamma \frac{\delta^2 W}{\delta \beta\, \delta \gamma} &= 0; \\
\frac{\delta W}{\delta \gamma} + \alpha \frac{\delta^2 W}{\delta \alpha\, \delta \gamma} + \beta \frac{\delta^2 W}{\delta \beta\, \delta \gamma} + \gamma \frac{\delta^2 W}{\delta \gamma^2} &= 0; \\
2 \frac{\delta^2 W}{\delta \alpha^2} + \alpha \frac{\delta^3 W}{\delta \alpha^3} + \beta \frac{\delta^3 W}{\delta \alpha^2\, \delta \beta} + \gamma \frac{\delta^3 W}{\delta \alpha^2\, \delta \gamma} &= 0; \\
\&\text{c.} &
\end{aligned}\right\} \qquad \text{(F)}$$

We have also, by the homogeneous nature of v, which we have put under the form of a function of the first dimension, the following relations:

$$\left.\begin{aligned}
\alpha\frac{\delta v}{\delta\alpha}+\beta\frac{\delta v}{\delta\beta}+\gamma\frac{\delta v}{\delta\gamma}&=v;\\
\alpha\frac{\delta^2 v}{\delta\alpha^2}+\beta\frac{\delta^2 v}{\delta\alpha\,\delta\beta}+\gamma\frac{\delta^2 v}{\delta\alpha\,\delta\gamma}&=0;\\
\alpha\frac{\delta^2 v}{\delta\alpha\,\delta\beta}+\beta\frac{\delta^2 v}{\delta\beta^2}+\gamma\frac{\delta^2 v}{\delta\beta\,\delta\gamma}&=0;\\
\alpha\frac{\delta^2 v}{\delta\alpha\,\delta\gamma}+\beta\frac{\delta^2 v}{\delta\beta\,\delta\gamma}+\gamma\frac{\delta^2 v}{\delta\gamma^2}&=0;\\
\frac{\delta^2 v}{\delta\alpha^2}+\alpha\frac{\delta^3 v}{\delta\alpha^3}+\beta\frac{\delta^3 v}{\delta\alpha^2\,\delta\beta}+\gamma\frac{\delta^3 v}{\delta\alpha^2\,\delta\gamma}&=0;\\
\&\text{c.}
\end{aligned}\right\}\tag{G}$$

These relations give

$$\alpha\,\delta\frac{\delta v}{\delta\alpha}+\beta\,\delta\frac{\delta v}{\delta\beta}+\gamma\,\delta\frac{\delta v}{\delta\gamma}=0,$$

and therefore, by (C),

$$\alpha\,\delta\frac{\delta V}{\delta x}+\beta\,\delta\frac{\delta V}{\delta y}+\gamma\,\delta\frac{\delta V}{\delta z}=0,$$

a condition which resolves itself into the three following,

$$\left.\begin{aligned}
\alpha\frac{\delta^2 V}{\delta x^2}+\beta\frac{\delta^2 V}{\delta x\,\delta y}+\gamma\frac{\delta^2 V}{\delta x\,\delta z}&=0;\\
\alpha\frac{\delta^2 V}{\delta x\,\delta y}+\beta\frac{\delta^2 V}{\delta y^2}+\gamma\frac{\delta^2 V}{\delta y\,\delta z}&=0;\\
\alpha\frac{\delta^2 V}{\delta x\,\delta z}+\beta\frac{\delta^2 V}{\delta y\,\delta z}+\gamma\frac{\delta^2 V}{\delta z^2}&=0;
\end{aligned}\right\}\tag{H}$$

and combining these three equations (H) with those which are obtained by differentiating (C), we find,

$$\left.\begin{aligned}
V''\,\delta x'&=\left(\frac{\delta^2 V}{\delta x^2}+\frac{\delta^2 V}{\delta y^2}+\frac{\delta^2 V}{\delta z^2}\right)\delta\frac{\delta v}{\delta\alpha}-\left(\frac{\delta^2 V}{\delta x^2}\,\delta\frac{\delta v}{\delta\alpha}+\frac{\delta^2 V}{\delta x\,\delta y}\,\delta\frac{\delta v}{\delta\beta}+\frac{\delta^2 V}{\delta x\,\delta z}\,\delta\frac{\delta v}{\delta\gamma}\right),\\
V''\,\delta y'&=\left(\frac{\delta^2 V}{\delta x^2}+\frac{\delta^2 V}{\delta y^2}+\frac{\delta^2 V}{\delta z^2}\right)\delta\frac{\delta v}{\delta\beta}-\left(\frac{\delta^2 V}{\delta x\,\delta y}\,\delta\frac{\delta v}{\delta\alpha}+\frac{\delta^2 V}{\delta y^2}\,\delta\frac{\delta v}{\delta\beta}+\frac{\delta^2 V}{\delta y\,\delta z}\,\delta\frac{\delta v}{\delta\gamma}\right),\\
V''\,\delta z'&=\left(\frac{\delta^2 V}{\delta x^2}+\frac{\delta^2 V}{\delta y^2}+\frac{\delta^2 V}{\delta z^2}\right)\delta\frac{\delta v}{\delta\gamma}-\left(\frac{\delta^2 V}{\delta x\,\delta z}\,\delta\frac{\delta v}{\delta\alpha}+\frac{\delta^2 V}{\delta y\,\delta z}\,\delta\frac{\delta v}{\delta\beta}+\frac{\delta^2 V}{\delta z^2}\,\delta\frac{\delta v}{\delta\gamma}\right),
\end{aligned}\right\}\tag{I}$$

in which

$$V''=\frac{\delta^2 V}{\delta x^2}\frac{\delta^2 V}{\delta y^2}-\left(\frac{\delta^2 V}{\delta x\,\delta y}\right)^2+\frac{\delta^2 V}{\delta y^2}\frac{\delta^2 V}{\delta z^2}-\left(\frac{\delta^2 V}{\delta y\,\delta z}\right)^2+\frac{\delta^2 V}{\delta z^2}\frac{\delta^2 V}{\delta x^2}-\left(\frac{\delta^2 V}{\delta z\,\delta x}\right)^2,$$

and

$$\delta x' = \delta x - \alpha(\alpha\,\delta x + \beta\,\delta y + \gamma\,\delta z),$$
$$\delta y' = \delta y - \beta(\alpha\,\delta x + \beta\,\delta y + \gamma\,\delta z),$$
$$\delta z' = \delta z - \gamma(\alpha\,\delta x + \beta\,\delta y + \gamma\,\delta z);$$

so that

$$\alpha\,\delta x' + \beta\,\delta y' + \gamma\,\delta z' = 0.$$

Now, if we differentiate the expressions,

$$\left.\begin{aligned}
\frac{\delta W}{\delta\alpha} &= x\frac{\delta^2 v}{\delta\alpha^2} + y\frac{\delta^2 v}{\delta\alpha\,\delta\beta} + z\frac{\delta^2 v}{\delta\alpha\,\delta\gamma},\\
\frac{\delta W}{\delta\beta} &= x\frac{\delta^2 v}{\delta\alpha\,\delta\beta} + y\frac{\delta^2 v}{\delta\beta^2} + z\frac{\delta^2 v}{\delta\beta\,\delta\gamma},\\
\frac{\delta W}{\delta\gamma} &= x\frac{\delta^2 v}{\delta\alpha\,\delta\gamma} + y\frac{\delta^2 v}{\delta\beta\,\delta\gamma} + z\frac{\delta^2 v}{\delta\gamma^2},
\end{aligned}\right\} \tag{K}$$

which result from the foregoing number, and put for abridgment,

$$\begin{aligned}
\delta\alpha\,.\,\delta\frac{\delta W}{\delta\alpha} + \delta\beta\,.\,\delta\frac{\delta W}{\delta\beta} + \delta\gamma\,.\,\delta\frac{\delta W}{\delta\gamma} &= \delta^2 W,\\
\delta\alpha\,.\,\delta\frac{\delta^2 v}{\delta\alpha^2} + \delta\beta\,.\,\delta\frac{\delta^2 v}{\delta\alpha\,\delta\beta} + \delta\gamma\,.\,\delta\frac{\delta^2 v}{\delta\alpha\,\delta\gamma} &= \delta^2\frac{\delta v}{\delta\alpha},\\
\delta\alpha\,.\,\delta\frac{\delta^2 v}{\delta\alpha\,\delta\beta} + \delta\beta\,.\,\delta\frac{\delta^2 v}{\delta\beta^2} + \delta\gamma\,.\,\delta\frac{\delta^2 v}{\delta\beta\,\delta\gamma} &= \delta^2\frac{\delta v}{\delta\beta},\\
\delta\alpha\,.\,\delta\frac{\delta^2 v}{\delta\alpha\,\delta\gamma} + \delta\beta\,.\,\delta\frac{\delta^2 v}{\delta\beta\,\delta\gamma} + \delta\gamma\,.\,\delta\frac{\delta^2 v}{\delta\gamma^2} &= \delta^2\frac{\delta v}{\delta\gamma},
\end{aligned}$$

$$\begin{aligned}
\delta x\,.\,\frac{\delta^2 v}{\delta\alpha^2} + \delta y\,.\,\frac{\delta^2 v}{\delta\alpha\,\delta\beta} + \delta z\,.\,\frac{\delta^2 v}{\delta\alpha\,\delta\gamma} &= \delta'\frac{\delta v}{\delta\alpha},\\
\delta x\,.\,\frac{\delta^2 v}{\delta\alpha\,\delta\beta} + \delta y\,.\,\frac{\delta^2 v}{\delta\beta^2} + \delta z\,.\,\frac{\delta^2 v}{\delta\beta\,\delta\gamma} &= \delta'\frac{\delta v}{\delta\beta},\\
\delta x\,.\,\frac{\delta^2 v}{\delta\alpha\,\delta\gamma} + \delta y\,.\,\frac{\delta^2 v}{\delta\beta\,\delta\gamma} + \delta z\,.\,\frac{\delta^2 v}{\delta\gamma^2} &= \delta'\frac{\delta v}{\delta\gamma},
\end{aligned}$$

we find

$$\begin{aligned}
\delta^2 W - \left(x\,\delta^2\frac{\delta v}{\delta\alpha} + y\,\delta^2\frac{\delta v}{\delta\beta} + z\,\delta^2\frac{\delta v}{\delta\gamma}\right) &= \delta\alpha\,\delta'\frac{\delta v}{\delta\alpha} + \delta\beta\,\delta'\frac{\delta v}{\delta\beta} + \delta\gamma\,\delta'\frac{\delta v}{\delta\gamma},\\
&= \delta x'\,\delta\frac{\delta v}{\delta\alpha} + \delta y'\,\delta\frac{\delta v}{\delta\beta} + \delta z'\,\delta\frac{\delta v}{\delta\gamma},
\end{aligned}$$

and therefore, by (I),

$$V''\,\delta^2 W = V''\left(x\,\delta^2\frac{\delta v}{\delta\alpha} + y\,\delta^2\frac{\delta v}{\delta\beta} + z\,\delta^2\frac{\delta v}{\delta\gamma}\right)$$

$$+\left(\frac{\delta^2 V}{\delta x^2}+\frac{\delta^2 V}{\delta y^2}+\frac{\delta^2 V}{\delta z^2}\right)\left\{\left(\delta\frac{\delta v}{\delta\alpha}\right)^2+\left(\delta\frac{\delta v}{\delta\beta}\right)^2+\left(\delta\frac{\delta v}{\delta\gamma}\right)^2\right\}$$
$$-\left\{\frac{\delta^2 V}{\delta x^2}\left(\delta\frac{\delta v}{\delta\alpha}\right)^2+\frac{\delta^2 V}{\delta y^2}\left(\delta\frac{\delta v}{\delta\beta}\right)^2+\frac{\delta^2 V}{\delta z^2}\left(\delta\frac{\delta v}{\delta\gamma}\right)^2+2\frac{\delta^2 V}{\delta x\,\delta y}\left(\delta\frac{\delta v}{\delta\alpha}\right)\left(\delta\frac{\delta v}{\delta\beta}\right)\right.$$
$$\left.+2\frac{\delta^2 V}{\delta y\,\delta z}\left(\delta\frac{\delta v}{\delta\beta}\right)\left(\delta\frac{\delta v}{\delta\gamma}\right)+2\frac{\delta^2 V}{\delta z\,\delta x}\left(\delta\frac{\delta v}{\delta\gamma}\right)\left(\delta\frac{\delta v}{\delta\alpha}\right)\right\}, \tag{L}$$

in which, without violating the conditions (F), the variations $\delta\alpha$, $\delta\beta$, $\delta\gamma$, may be considered as independent, and which is consequently equivalent to six expressions for the six partial differential coefficients of W, of the second order.

These six expressions may be put under the following form:

$$\left.\begin{aligned}
\frac{\delta^2 W}{\delta\alpha^2} &= x\frac{\delta^3 v}{\delta\alpha^3}+y\frac{\delta^3 v}{\delta\alpha^2\,\delta\beta}+z\frac{\delta^3 v}{\delta\alpha^2\,\delta\gamma}+\frac{S}{V''}\frac{\delta^2 v}{\delta\alpha^2}-\frac{v''}{V''}\frac{\delta^2 V}{\delta x^2},\\
\frac{\delta^2 W}{\delta\beta^2} &= x\frac{\delta^3 v}{\delta\alpha\,\delta\beta^2}+y\frac{\delta^3 v}{\delta\beta^3}+z\frac{\delta^3 v}{\delta\beta^2\,\delta\gamma}+\frac{S}{V''}\frac{\delta^2 v}{\delta\beta^2}-\frac{v''}{V''}\frac{\delta^2 V}{\delta y^2},\\
\frac{\delta^2 W}{\delta\gamma^2} &= x\frac{\delta^3 v}{\delta\alpha\,\delta\gamma^2}+y\frac{\delta^3 v}{\delta\beta\,\delta\gamma^2}+z\frac{\delta^3 v}{\delta\gamma^3}+\frac{S}{V''}\frac{\delta^2 v}{\delta\gamma^2}-\frac{v''}{V''}\frac{\delta^2 V}{\delta z^2},\\
\frac{\delta^2 W}{\delta\alpha\,\delta\beta} &= x\frac{\delta^3 v}{\delta\alpha^2\,\delta\beta}+y\frac{\delta^3 v}{\delta\alpha\,\delta\beta^2}+z\frac{\delta^3 v}{\delta\alpha\,\delta\beta\,\delta\gamma}+\frac{S}{V''}\frac{\delta^2 v}{\delta\alpha\,\delta\beta}-\frac{v''}{V''}\frac{\delta^2 V}{\delta x\,\delta y},\\
\frac{\delta^2 W}{\delta\beta\,\delta\gamma} &= x\frac{\delta^3 v}{\delta\alpha\,\delta\beta\,\delta\gamma}+y\frac{\delta^3 v}{\delta\beta^2\,\delta\gamma}+z\frac{\delta^3 v}{\delta\beta\,\delta\gamma^2}+\frac{S}{V''}\frac{\delta^2 v}{\delta\beta\,\delta\gamma}-\frac{v''}{V''}\frac{\delta^2 V}{\delta y\,\delta z},\\
\frac{\delta^2 W}{\delta\gamma\,\delta\alpha} &= x\frac{\delta^3 v}{\delta\alpha^2\,\delta\gamma}+y\frac{\delta^3 v}{\delta\alpha\,\delta\beta\,\delta\gamma}+z\frac{\delta^3 v}{\delta\alpha\,\delta\gamma^2}+\frac{S}{V''}\frac{\delta^2 v}{\delta\gamma\,\delta\alpha}-\frac{v''}{V''}\frac{\delta^2 V}{\delta z\,\delta x};
\end{aligned}\right\} \tag{M}$$

in which

$$v''=\frac{\delta^2 v}{\delta\alpha^2}\frac{\delta^2 v}{\delta\beta^2}-\left(\frac{\delta^2 v}{\delta\alpha\,\delta\beta}\right)^2+\frac{\delta^2 v}{\delta\beta^2}\frac{\delta^2 v}{\delta\gamma^2}-\left(\frac{\delta^2 v}{\delta\beta\,\delta\gamma}\right)^2+\frac{\delta^2 v}{\delta\gamma^2}\frac{\delta^2 v}{\delta\alpha^2}-\left(\frac{\delta^2 v}{\delta\gamma\,\delta\alpha}\right)^2,$$

and

$$S=\left(\frac{\delta^2 v}{\delta\alpha^2}+\frac{\delta^2 v}{\delta\beta^2}+\frac{\delta^2 v}{\delta\gamma^2}\right)\left(\frac{\delta^2 V}{\delta x^2}+\frac{\delta^2 V}{\delta y^2}+\frac{\delta^2 V}{\delta z^2}\right)$$
$$-\left(\frac{\delta^2 v}{\delta\alpha^2}\frac{\delta^2 V}{\delta x^2}+\frac{\delta^2 v}{\delta\beta^2}\frac{\delta^2 V}{\delta y^2}+\frac{\delta^2 v}{\delta\gamma^2}\frac{\delta^2 V}{\delta z^2}+2\frac{\delta^2 v}{\delta\alpha\,\delta\beta}\frac{\delta^2 V}{\delta x\,\delta y}+2\frac{\delta^2 v}{\delta\beta\,\delta\gamma}\frac{\delta^2 V}{\delta y\,\delta z}+2\frac{\delta^2 v}{\delta\gamma\,\delta\alpha}\frac{\delta^2 V}{\delta z\,\delta x}\right).$$

These expressions enable us to deduce the partial differential coefficients of W, of the second order, from the corresponding differentials of V; they may also be employed to deduce the latter from the former. For if we put

$$\frac{\delta^2 W}{\delta\alpha^2} - \left(x\frac{\delta^3 v}{\delta\alpha^3} + y\frac{\delta^3 v}{\delta\alpha^2\,\delta\beta} + z\frac{\delta^3 v}{\delta\alpha^2\,\delta\gamma}\right) = M,$$
$$\frac{\delta^2 W}{\delta\beta^2} - \left(x\frac{\delta^3 v}{\delta\alpha\,\delta\beta^2} + y\frac{\delta^3 v}{\delta\beta^3} + z\frac{\delta^3 v}{\delta\beta^2\,\delta\gamma}\right) = N,$$
$$\frac{\delta^2 W}{\delta\gamma^2} - \left(x\frac{\delta^3 v}{\delta\alpha\,\delta\gamma^2} + y\frac{\delta^3 v}{\delta\beta\,\delta\gamma^2} + z\frac{\delta^3 v}{\delta\gamma^3}\right) = P,$$
$$\frac{\delta^2 W}{\delta\alpha\,\delta\beta} - \left(x\frac{\delta^3 v}{\delta\alpha^2\,\delta\beta} + y\frac{\delta^3 v}{\delta\alpha\,\delta\beta^2} + z\frac{\delta^3 v}{\delta\alpha\,\delta\beta\,\delta\gamma}\right) = M',$$
$$\frac{\delta^2 W}{\delta\beta\,\delta\gamma} - \left(x\frac{\delta^3 v}{\delta\alpha\,\delta\beta\,\delta\gamma} + y\frac{\delta^3 v}{\delta\beta^2\,\delta\gamma} + z\frac{\delta^3 v}{\delta\beta\,\delta\gamma^2}\right) = N',$$
$$\frac{\delta^2 W}{\delta\gamma\,\delta\alpha} - \left(x\frac{\delta^3 v}{\delta\alpha^2\,\delta\gamma} + y\frac{\delta^3 v}{\delta\alpha\,\delta\beta\,\delta\gamma} + z\frac{\delta^3 v}{\delta\alpha\,\delta\gamma^2}\right) = P',$$

$$MN - M'^2 + NP - N'^2 + PM - P'^2 = W'',$$
$$(M+N+P)\left(\frac{\delta^2 v}{\delta\alpha^2} + \frac{\delta^2 v}{\delta\beta^2} + \frac{\delta^2 v}{\delta\gamma^2}\right) - \left(M\frac{\delta^2 v}{\delta\alpha^2} + N\frac{\delta^2 v}{\delta\beta^2} + P\frac{\delta^2 v}{\delta\gamma^2}\right.$$
$$\left. +2M'\frac{\delta^2 v}{\delta\alpha\,\delta\beta} + 2N'\frac{\delta^2 v}{\delta\beta\,\delta\gamma} + 2P'\frac{\delta^2 v}{\delta\gamma\,\delta\alpha}\right) = S',$$

we find, by the equations (M),

$$V''W'' = v''^2, \quad V''S' = v''S : \tag{N}$$

and therefore

$$\left.\begin{aligned}
\frac{d^2 V}{\delta x^2} &= \frac{S'}{W''}\frac{\delta^2 v}{\delta\alpha^2} - \frac{v''M}{W''}, & \frac{d^2 V}{\delta x\,\delta y} &= \frac{S'}{W''}\frac{\delta^2 v}{\delta\alpha\,\delta\beta} - \frac{v''M'}{W''},\\
\frac{d^2 V}{\delta y^2} &= \frac{S'}{W''}\frac{\delta^2 v}{\delta\beta^2} - \frac{v''N}{W''}, & \frac{d^2 V}{\delta y\,\delta z} &= \frac{S'}{W''}\frac{\delta^2 v}{\delta\beta\,\delta\gamma} - \frac{v''N'}{W''},\\
\frac{d^2 V}{\delta z^2} &= \frac{S'}{W''}\frac{\delta^2 v}{\delta\gamma^2} - \frac{v''P}{W''}, & \frac{d^2 V}{\delta z\,\delta x} &= \frac{S'}{W''}\frac{\delta^2 v}{\delta\gamma\,\delta\alpha} - \frac{v''P'}{W''}.
\end{aligned}\right\} \tag{O}$$

The coefficients of the form $\frac{\delta^2 V}{\delta x^2}$, may also be deduced from those of the form $\frac{\delta^2 W}{\delta\alpha^2}$, in the following manner. Differentiating the equations (K) we obtain

$$\left.\begin{aligned}
M\,\delta\alpha + M'\,\delta\beta + P'\,\delta\gamma &= \delta'\frac{\delta v}{\delta\alpha},\\
M'\,\delta\alpha + N\,\delta\beta + N'\,\delta\gamma &= \delta'\frac{\delta v}{\delta\beta},\\
P'\,\delta\alpha + N'\,\delta\beta + P\,\delta\gamma &= \delta'\frac{\delta v}{\delta\gamma};
\end{aligned}\right\} \tag{P}$$

we have also

$$
\begin{aligned}
0 &= \alpha M + \beta M' + \gamma P', \\
0 &= \alpha M' + \beta N + \gamma N', \\
0 &= \alpha P' + \beta N' + \gamma P;
\end{aligned}
$$

and therefore

$$
\left.
\begin{aligned}
W''\,\delta\alpha &= (M + N + P)\delta' \frac{\delta v}{\delta\alpha} - \left(M\,\delta' \frac{\delta v}{\delta\alpha} + M'\,\delta' \frac{\delta v}{\delta\beta} + P'\,\delta' \frac{\delta v}{\delta\gamma} \right), \\
W''\,\delta\beta &= (M + N + P)\delta' \frac{\delta v}{\delta\beta} - \left(M'\,\delta' \frac{\delta v}{\delta\alpha} + N\,\delta' \frac{\delta v}{\delta\beta} + N'\,\delta' \frac{\delta v}{\delta\gamma} \right), \\
W''\,\delta\gamma &= (M + N + P)\delta' \frac{\delta v}{\delta\gamma} - \left(P'\,\delta' \frac{\delta v}{\delta\alpha} + N'\,\delta' \frac{\delta v}{\delta\beta} + P\,\delta' \frac{\delta v}{\delta\gamma} \right).
\end{aligned}
\right\} \tag{Q}
$$

Now, if we put

$$
\delta^2 V = \delta x\,\delta \frac{\delta V}{\delta x} + \delta y\,\delta \frac{\delta V}{\delta y} + \delta z\,\delta \frac{\delta V}{\delta z},
$$

we shall have

$$
\delta^2 V = \delta\alpha\,\delta' \frac{\delta v}{\delta\alpha} + \delta\beta\,\delta' \frac{\delta v}{\delta\beta} + \delta\gamma\,\delta' \frac{\delta v}{\delta\gamma},
$$

and therefore by (Q),

$$
\begin{aligned}
W''\,\delta^2 V = (M + N + P) &\left\{ \left(\delta' \frac{\delta v}{\delta\alpha} \right)^2 + \left(\delta' \frac{\delta v}{\delta\beta} \right)^2 + \left(\delta' \frac{\delta v}{\delta\gamma} \right)^2 \right\} \\
&- \left\{ M \left(\delta' \frac{\delta v}{\delta\alpha} \right)^2 + N \left(\delta' \frac{\delta v}{\delta\beta} \right)^2 + P \left(\delta' \frac{\delta v}{\delta\gamma} \right)^2 \right. \\
&\left. + 2M' \left(\delta' \frac{\delta v}{\delta\alpha} \right) \left(\delta' \frac{\delta v}{\delta\beta} \right) + 2N' \left(\delta' \frac{\delta v}{\delta\beta} \right) \left(\delta' \frac{\delta v}{\delta\gamma} \right) + 2P' \left(\delta' \frac{\delta v}{\delta\gamma} \right) \left(\delta' \frac{\delta v}{\delta\alpha} \right) \right\}; \text{(R)}
\end{aligned}
$$

an equation in which δx, δy, δz, are independent, so that it is equivalent to six separate expressions, for the six partial differential coefficients of V, of the second order: and these expressions may easily be shewn to coincide with those marked (O). And on similar principles we can investigate the relations between the partial differential coefficients of the functions V and W, for the third and higher orders.

Changes produced by Reflexions or Refractions, Ordinary or Extraordinary.

6. Let us now consider the sudden changes in these partial differential coefficients of the characteristic functions of the system, produced by reflexion or refraction, ordinary or extraordinary. The general formula for such changes, is, from the nature of the integral V,

$$
\Delta V\,(= V_2 - V_1) = 0, \tag{S}
$$

Δ being here the symbol of a finite difference, and V_1, V_2, being two successive forms of the function V, before and after reflexion or refraction. The condition (S) may be considered as a form of the equation of the reflecting or refracting surface; and if $u = 0$ be any other form for the equation of this surface, we may, by introducing a multiplier λ, differentiate the following formula:

$$\Delta V \, (= V_2 - V_1) = \lambda u, \tag{T}$$

as if the coordinates x, y, z, were three independent variables. Differentiating in this manner the equation (T), and making, after the differentiations, $u = 0$, we find

$$\left.\begin{aligned}
\Delta \frac{\delta V}{\delta x} &= \frac{\delta V_2}{\delta x} - \frac{\delta V_1}{\delta x} = \lambda \frac{\delta u}{\delta x}, \\
\Delta \frac{\delta V}{\delta y} &= \frac{\delta V_2}{\delta y} - \frac{\delta V_1}{\delta y} = \lambda \frac{\delta u}{\delta y}, \\
\Delta \frac{\delta V}{\delta z} &= \frac{\delta V_2}{\delta z} - \frac{\delta V_1}{\delta z} = \lambda \frac{\delta u}{\delta z};
\end{aligned}\right\} \tag{U}$$

$$\left.\begin{aligned}
\Delta \frac{\delta^2 V}{\delta x^2} &= \frac{\delta^2 V_2}{\delta x^2} - \frac{\delta^2 V_1}{\delta x^2} = \lambda \frac{\delta^2 u}{\delta x^2} + 2 \frac{d\lambda}{\delta x} \frac{\delta u}{\delta x}, \\
\Delta \frac{\delta^2 V}{\delta y^2} &= \frac{\delta^2 V_2}{\delta y^2} - \frac{\delta^2 V_1}{\delta y^2} = \lambda \frac{\delta^2 u}{\delta y^2} + 2 \frac{d\lambda}{\delta y} \frac{\delta u}{\delta y}, \\
\Delta \frac{\delta^2 V}{\delta z^2} &= \frac{\delta^2 V_2}{\delta z^2} - \frac{\delta^2 V_1}{\delta z^2} = \lambda \frac{\delta^2 u}{\delta z^2} + 2 \frac{d\lambda}{\delta z} \frac{\delta u}{\delta z}, \\
\Delta \frac{\delta^2 V}{\delta x \, \delta y} &= \frac{\delta^2 V_2}{\delta x \, \delta y} - \frac{\delta^2 V_1}{\delta x \, \delta y} = \lambda \frac{\delta^2 u}{\delta x \, \delta y} + \frac{d\lambda}{\delta x} \frac{\delta u}{\delta y} + \frac{d\lambda}{\delta y} \frac{\delta u}{\delta x}, \\
\Delta \frac{\delta^2 V}{\delta y \, \delta z} &= \frac{\delta^2 V_2}{\delta y \, \delta z} - \frac{\delta^2 V_1}{\delta y \, \delta z} = \lambda \frac{\delta^2 u}{\delta y \, \delta z} + \frac{d\lambda}{\delta y} \frac{\delta u}{\delta z} + \frac{d\lambda}{\delta z} \frac{\delta u}{\delta y}, \\
\Delta \frac{\delta^2 V}{\delta z \, \delta x} &= \frac{\delta^2 V_2}{\delta z \, \delta x} - \frac{\delta^2 V_1}{\delta z \, \delta x} = \lambda \frac{\delta^2 u}{\delta z \, \delta x} + \frac{d\lambda}{\delta z} \frac{\delta u}{\delta x} + \frac{d\lambda}{\delta x} \frac{\delta u}{\delta z}.
\end{aligned}\right\} \tag{V}$$

The equations marked (U), contain the laws of reflexion and refraction, ordinary and extraordinary; since, when put by means of (C) under the form

$$\left.\begin{aligned}
\Delta \frac{\delta v}{\delta \alpha} &= \frac{\delta v_2}{\delta \alpha_2} - \frac{\delta v_1}{\delta \alpha_1} = \lambda \frac{\delta u}{\delta x}, \\
\Delta \frac{\delta v}{\delta \beta} &= \frac{\delta v_2}{\delta \beta_2} - \frac{\delta v_1}{\delta \beta_1} = \lambda \frac{\delta u}{\delta y}, \\
\Delta \frac{\delta v}{\delta \gamma} &= \frac{\delta v_2}{\delta \gamma_2} - \frac{\delta v_1}{\delta \gamma_1} = \lambda \frac{\delta u}{\delta z},
\end{aligned}\right\} \tag{W}$$

and combined with the relation $\alpha_2^2 + \beta_2^2 + \gamma_2^2 = 1$, they suffice to determine, for any given forms of the functions v_1, v_2, and for any given directions of the incident ray and of the tangent plane to the reflecting or refracting surface, the cosines α_2, β_2, γ_2, of the angles which the reflected or refracted ray makes with the axes of coordinates, and the value of the multiplier λ; observing that the ratio

$$\frac{\alpha_2 \left(\dfrac{\delta u}{\delta x}\right) + \beta_2 \left(\dfrac{\delta u}{\delta y}\right) + \gamma_2 \left(\dfrac{\delta u}{\delta z}\right)}{\alpha_1 \left(\dfrac{\delta u}{\delta x}\right) + \beta_1 \left(\dfrac{\delta u}{\delta y}\right) + \gamma_1 \left(\dfrac{\delta u}{\delta z}\right)},$$

is positive in the case of refraction, and negative in that of reflexion. The equations (V), when combined with the relations (H), determine the six partial differential coefficients of V_2 of the second order, together with the three quantities

$$\frac{\delta \lambda}{\delta x}, \quad \frac{\delta \lambda}{\delta y}, \quad \frac{\delta \lambda}{\delta z};$$

since they give, for these three latter quantities, the conditions

$$\left.\begin{aligned}
0 &= \alpha_2 \left(\frac{\delta^2 V_1}{\delta x^2} + \lambda \frac{\delta^2 u}{\delta x^2}\right) + \beta_2 \left(\frac{\delta^2 V_1}{\delta x\, \delta y} + \lambda \frac{\delta^2 u}{\delta x\, \delta y}\right) + \gamma_2 \left(\frac{\delta^2 V_1}{\delta x\, \delta z} + \lambda \frac{\delta^2 u}{\delta x\, \delta z}\right) \\
&\quad + \frac{\delta \lambda}{\delta x} \left(\alpha_2 \frac{\delta u}{\delta x} + \beta_2 \frac{\delta u}{\delta y} + \gamma_2 \frac{\delta u}{\delta z}\right) + \frac{\delta u}{\delta x} \left(\alpha_2 \frac{\delta \lambda}{\delta x} + \beta_2 \frac{\delta \lambda}{\delta y} + \gamma_2 \frac{\delta \lambda}{\delta z}\right); \\
0 &= \alpha_2 \left(\frac{\delta^2 V_1}{\delta x\, \delta y} + \lambda \frac{\delta^2 u}{\delta x\, \delta y}\right) + \beta_2 \left(\frac{\delta^2 V_1}{\delta y^2} + \lambda \frac{\delta^2 u}{\delta y^2}\right) + \gamma_2 \left(\frac{\delta^2 V_1}{\delta y\, \delta z} + \lambda \frac{\delta^2 u}{\delta y\, \delta z}\right) \\
&\quad + \frac{\delta \lambda}{\delta y} \left(\alpha_2 \frac{\delta u}{\delta x} + \beta_2 \frac{\delta u}{\delta y} + \gamma_2 \frac{\delta u}{\delta z}\right) + \frac{\delta u}{\delta y} \left(\alpha_2 \frac{\delta \lambda}{\delta x} + \beta_2 \frac{\delta \lambda}{\delta y} + \gamma_2 \frac{\delta \lambda}{\delta z}\right); \\
0 &= \alpha_2 \left(\frac{\delta^2 V_1}{\delta x\, \delta z} + \lambda \frac{\delta^2 u}{\delta x\, \delta z}\right) + \beta_2 \left(\frac{\delta^2 V_1}{\delta y\, \delta z} + \lambda \frac{\delta^2 u}{\delta y\, \delta z}\right) + \gamma_2 \left(\frac{\delta^2 V_1}{\delta z^2} + \lambda \frac{\delta^2 u}{\delta z^2}\right) \\
&\quad + \frac{\delta \lambda}{\delta z} \left(\alpha_2 \frac{\delta u}{\delta x} + \beta_2 \frac{\delta u}{\delta y} + \gamma_2 \frac{\delta u}{\delta z}\right) + \frac{\delta u}{\delta z} \left(\alpha_2 \frac{\delta \lambda}{\delta x} + \beta_2 \frac{\delta \lambda}{\delta y} + \gamma_2 \frac{\delta \lambda}{\delta z}\right):
\end{aligned}\right\} \quad \text{(X)}$$

in which the trinomial

$$\left(\alpha_2 \frac{\delta \lambda}{\delta x} + \beta_2 \frac{\delta \lambda}{\delta y} + \gamma_2 \frac{\delta \lambda}{\delta z}\right)$$

can be determined by the following relation:

$$\begin{aligned}
0 &= \alpha_2^2 \left(\frac{\delta^2 V_1}{\delta x^2} + \lambda \frac{\delta^2 u}{\delta x^2}\right) + \beta_2^2 \left(\frac{\delta^2 V_1}{\delta y^2} + \lambda \frac{\delta^2 u}{\delta y^2}\right) + \gamma_2^2 \left(\frac{\delta^2 V_1}{\delta z^2} + \lambda \frac{\delta^2 u}{\delta z^2}\right) \\
&\quad + 2\alpha_2\beta_2 \left(\frac{\delta^2 V_1}{\delta x\, \delta y} + \lambda \frac{\delta^2 u}{\delta x\, \delta y}\right) + 2\beta_2\gamma_2 \left(\frac{\delta^2 V_1}{\delta y\, \delta z} + \lambda \frac{\delta^2 u}{\delta y\, \delta z}\right) + 2\gamma_2\alpha_2 \left(\frac{\delta^2 V_1}{\delta z\, \delta x} + \lambda \frac{\delta^2 u}{\delta z\, \delta x}\right) \\
&\quad + 2 \left(\alpha_2 \frac{\delta \lambda}{\delta x} + \beta_2 \frac{\delta \lambda}{\delta y} + \gamma_2 \frac{\delta \lambda}{\delta z}\right) \left(\alpha_2 \frac{\delta u}{\delta x} + \beta_2 \frac{\delta u}{\delta y} + \gamma_2 \frac{\delta u}{\delta z}\right).
\end{aligned}$$

In a similar manner we can calculate the new values which are given, by reflexion or refraction, to the partial differential coefficients of V, of the third and higher orders; and can thence deduce the corresponding changes in the coefficients of the function W, by means of the relations which we have already pointed out, between these two characteristic functions; observing, that while the value of V itself is not altered in the act of reflexion or refraction, but only its form and its differentials, the value of W receives a sudden increment, which has for expression,

$$\begin{aligned} \Delta W &= x\,\Delta\frac{\delta v}{\delta\alpha} + y\,\Delta\frac{\delta v}{\delta\beta} + z\,\Delta\frac{\delta v}{\delta\gamma} \\ &= \lambda\left(x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} + z\frac{\delta u}{\delta z}\right). \end{aligned} \tag{Y}$$

7. By the help of the foregoing formulæ, we can compute the partial differential coefficients of any given order, of the characteristic functions V and W, for any homogeneous system of straight rays, produced by any finite number of successive reflexions and refractions ordinary or extraordinary, when we know the nature of the light and of the mediums, and know also the coordinates of the luminous origin and the equations of the reflecting or refracting surfaces. To shew this more fully, let us observe, that in a system of straight rays diverging from a luminous point, and not yet reflected or refracted, we may put

$$x - X = \alpha\rho, \quad y - Y = \beta\rho, \quad z - Z = \gamma\rho,$$

ρ being the distance from the luminous origin X, Y, Z, to any other point x, y, z; and that we have the equations,

$$\left.\begin{aligned} V &= v\rho = (x - X)\frac{\delta v}{\delta\alpha} + (y - Y)\frac{\delta v}{\delta\beta} + (z - Z)\frac{\delta v}{\delta\gamma}, \\ W &= X\frac{\delta v}{\delta\alpha} + Y\frac{\delta v}{\delta\beta} + Z\frac{\delta v}{\delta\gamma}, \end{aligned}\right\} \tag{Z}$$

from which we can deduce the partial differentials of the functions V and W, in this first state of the system; those of the second order, for example, being given by the following expressions:

$$\begin{aligned} \rho\,\delta^2 V &= \delta x\,\delta'\frac{\delta v}{\delta\alpha} + \delta y\,\delta'\frac{\delta v}{\delta\beta} + \delta z\,\delta'\frac{\delta v}{\delta\gamma}, \\ \delta^2 W &= X\,\delta^2\frac{\delta v}{\delta\alpha} + Y\,\delta^2\frac{\delta v}{\delta\beta} + Z\,\delta^2\frac{\delta v}{\delta\gamma}, \end{aligned}$$

in which the symbols

$$\delta'\frac{\delta v}{\delta\alpha}, \quad \delta^2\frac{\delta v}{\delta\alpha},$$

have the same meanings as before. Knowing, in this manner, the differential coefficients of V, before the first reflexion or refraction, we can, by the method of the preceding number, calculate the corresponding coefficients of V, and thence of W, immediately after that change; the coefficients of W, thus deduced, will remain the same, in passing from the point of first reflexion or refraction to the second point at which the direction of the ray is altered, and, by the methods of the fifth number, we can deduce from these coefficients of W the corresponding coefficients of V, immediately before that second change; and so proceeding, we can calculate the alterations in the partial differentials of the two characteristic functions, produced by any finite number of successive reflexions or refractions.

Connexions of the two Characteristic Functions with the Developable Pencils and the Caustic Curves and Surfaces.

8. Let us now suppose these partial differentials known, and let us examine their connexion with the geometrical properties of the system. One of the most remarkable of these geometrical properties is, that the rays are in general tangents to *two series of caustic curves*, which are contained upon *two caustic surfaces*, and form the *arêtes de rebroussement* of two series of *developable pencils*; that is, two series of developable surfaces, composed by rays of the system: a property which was first discovered by MALUS, and to which I also had arrived in my own researches, before I was aware of the existence of his. To investigate the connexion of these curves and surfaces with the characteristic functions V and W, let us consider the conditions which must be satisfied, in order that a curve having for coordinates x'', y'' z'', should be touched by an infinite number of rays of the system. Let x, y, z, be the coordinates of any point on such a ray, and ρ its distance from the point of contact x'' y'' z'', in such a manner that we may put

$$x = x'' + \alpha\rho, \quad y = y'' + \beta\rho, \quad z = z'' + \gamma\rho,$$

and therefore

$$\delta\rho = \alpha(\delta x - \delta x'') + \beta(\delta y - \delta y'') + \gamma(\delta z - \delta z'') :$$

we shall then have

$$\left.\begin{aligned} \delta x' &= \delta x - \alpha(\alpha\,\delta x + \beta\,\delta y + \gamma\,\delta z) = \rho\,\delta\alpha, \\ \delta y' &= \delta y - \beta(\alpha\,\delta x + \beta\,\delta y + \gamma\,\delta z) = \rho\,\delta\beta, \\ \delta z' &= \delta z - \gamma(\alpha\,\delta x + \beta\,\delta y + \gamma\,\delta z) = \rho\,\delta\gamma, \end{aligned}\right\} \tag{A$'$}$$

assigning to $\delta x'$ $\delta y'$ $\delta z'$ the same meanings as in the fifth number, and observing that by the nature of x'' y'' z'', the variations $\delta x''$ $\delta y''$ $\delta z''$ are proportional to α, β, γ, so that

$$\begin{aligned} \delta x'' &= \alpha(\alpha\,\delta x'' + \beta\,\delta y'' + \gamma\,\delta z''), \\ \delta y'' &= \beta(\alpha\,\delta x'' + \beta\,\delta y'' + \gamma\,\delta z''), \\ \delta z'' &= \gamma(\alpha\,\delta x'' + \beta\,\delta y'' + \gamma\,\delta z''). \end{aligned}$$

The formulæ (A$'$) give

$$\left.\begin{aligned} \delta'\frac{\delta v}{\delta\alpha} &= \rho\,\delta\frac{\delta v}{\delta\alpha} = \rho\,\delta\frac{\delta V}{\delta x}, \\ \delta'\frac{\delta v}{\delta\beta} &= \rho\,\delta\frac{\delta v}{\delta\beta} = \rho\,\delta\frac{\delta V}{\delta y}, \\ \delta'\frac{\delta v}{\delta\gamma} &= \rho\,\delta\frac{\delta v}{\delta\gamma} = \rho\,\delta\frac{\delta V}{\delta z}, \end{aligned}\right\} \tag{B$'$}$$

$\delta'\dfrac{\delta v}{\delta\alpha}$ having the same meaning as in the fifth number: and these equations (B$'$) contain the whole theory of the developable pencils and of the caustic curves and surfaces. Putting them under the form,

$$\left.\begin{aligned}
0 &= \left(\rho\frac{\delta^2 V}{\delta x^2} - \frac{\delta^2 v}{\delta\alpha^2}\right)\delta x + \left(\rho\frac{\delta^2 V}{\delta x\,\delta y} - \frac{\delta^2 v}{\delta\alpha\,\delta\beta}\right)\delta y + \left(\rho\frac{\delta^2 V}{\delta x\,\delta z} - \frac{\delta^2 v}{\delta\alpha\,\delta\gamma}\right)\delta z,\\
0 &= \left(\rho\frac{\delta^2 V}{\delta x\,\delta y} - \frac{\delta^2 v}{\delta\alpha\,\delta\beta}\right)\delta x + \left(\rho\frac{\delta^2 V}{\delta y^2} - \frac{\delta^2 v}{\delta\beta^2}\right)\delta y + \left(\rho\frac{\delta^2 V}{\delta y\,\delta z} - \frac{\delta^2 v}{\delta\beta\,\delta\gamma}\right)\delta z,\\
0 &= \left(\rho\frac{\delta^2 V}{\delta x\,\delta z} - \frac{\delta^2 v}{\delta\alpha\,\delta\gamma}\right)\delta x + \left(\rho\frac{\delta^2 V}{\delta y\,\delta z} - \frac{\delta^2 v}{\delta\beta\,\delta\gamma}\right)\delta y + \left(\rho\frac{\delta^2 V}{\delta z^2} - \frac{\delta^2 v}{\delta\gamma^2}\right)\delta z,
\end{aligned}\right\} \tag{C$'$}$$

we find by eliminating the differentials, and attending to the relations (G), (H), the following quadratic equation

$$0 = \rho^2 V'' - \rho S + v'', \tag{D$'$}$$

which may also be thus transformed,

$$0 = \rho^2 v'' - \rho S' + W'' : \tag{E$'$}$$

the symbols v'', V'', W'', S, S', having the same meanings as in the fifth number. The form (D$'$) serves to connect the distance ρ with the function V, and the form (E$'$) with W. By either of these forms, we obtain in general two values of ρ, and therefore two points $x''\ y''\ z''$, which are the only points wherein the ray can touch a caustic curve: and the locus of the points thus obtained, composes the *two caustic surfaces*. The joint equation of these surfaces, in $x''\ y''\ z''$, may be found by eliminating α, β, γ, between the four following equations:

$$\left.\begin{aligned}
x'' &= x_{\prime} + \alpha(\alpha x'' + \beta y'' + \gamma z''),\\
y'' &= y_{\prime} + \beta(\alpha x'' + \beta y'' + \gamma z''),\\
z'' &= z_{\prime} + \gamma(\alpha x'' + \beta y'' + \gamma z''),\\
0 &= (\alpha x'' + \beta y'' + \gamma z'')^2 v'' + (\alpha x'' + \beta y'' + \gamma z'')S'_{\prime} + W''_{\prime};
\end{aligned}\right\} \tag{F$'$}$$

in which $S'_{\prime}$, $W''_{\prime}$, are formed from S', W'', by changing x, y, z, to $x_{\prime}$, $y_{\prime}$, $z_{\prime}$, these latter symbols being abridged expressions for the following quantities,

$$\begin{aligned}
x - \alpha(\alpha x + \beta y + \gamma z) &= x_{\prime},\\
y - \beta(\alpha x + \beta y + \gamma z) &= y_{\prime},\\
z - \gamma(\alpha x + \beta y + \gamma z) &= z_{\prime},
\end{aligned}$$

and being considered as functions of α, β, γ, determined by the conditions

$$\left.\begin{aligned}
0 &= \alpha x_{\prime} + \beta y_{\prime} + \gamma z_{\prime},\\
\frac{\delta W}{\delta\alpha} &= x_{\prime}\frac{\delta^2 v}{\delta\alpha^2} + y_{\prime}\frac{\delta^2 v}{\delta\alpha\,\delta\beta} + z_{\prime}\frac{\delta^2 v}{\delta\alpha\,\delta\gamma},\\
\frac{\delta W}{\delta\beta} &= x_{\prime}\frac{\delta^2 v}{\delta\alpha\,\delta\beta} + y_{\prime}\frac{\delta^2 v}{\delta\beta^2} + z_{\prime}\frac{\delta^2 v}{\delta\beta\,\delta\gamma},\\
\frac{\delta W}{\delta\gamma} &= x_{\prime}\frac{\delta^2 v}{\delta\alpha\,\delta\gamma} + y_{\prime}\frac{\delta^2 v}{\delta\beta\,\delta\gamma} + z_{\prime}\frac{\delta^2 v}{\delta\gamma^2};
\end{aligned}\right\} \tag{G$'$}$$

which give, after elimination,

$$\left.\begin{aligned}
v''x_{\prime} &= \left(\frac{\delta^2 v}{\delta\alpha^2} + \frac{\delta^2 v}{\delta\beta^2} + \frac{\delta^2 v}{\delta\gamma^2}\right)\frac{\delta W}{\delta\alpha} - \left(\frac{\delta^2 v}{\delta\alpha^2}\frac{\delta W}{\delta\alpha} + \frac{\delta^2 v}{\delta\alpha\,\delta\beta}\frac{\delta W}{\delta\beta} + \frac{\delta^2 v}{\delta\alpha\,\delta\gamma}\frac{\delta W}{\delta\gamma}\right),\\
v''y_{\prime} &= \left(\frac{\delta^2 v}{\delta\alpha^2} + \frac{\delta^2 v}{\delta\beta^2} + \frac{\delta^2 v}{\delta\gamma^2}\right)\frac{\delta W}{\delta\beta} - \left(\frac{\delta^2 v}{\delta\alpha\,\delta\beta}\frac{\delta W}{\delta\alpha} + \frac{\delta^2 v}{\delta\beta^2}\frac{\delta W}{\delta\beta} + \frac{\delta^2 v}{\delta\beta\,\delta\gamma}\frac{\delta W}{\delta\gamma}\right),\\
v''z_{\prime} &= \left(\frac{\delta^2 v}{\delta\alpha^2} + \frac{\delta^2 v}{\delta\beta^2} + \frac{\delta^2 v}{\delta\gamma^2}\right)\frac{\delta W}{\delta\gamma} - \left(\frac{\delta^2 v}{\delta\alpha\,\delta\gamma}\frac{\delta W}{\delta\alpha} + \frac{\delta^2 v}{\delta\beta\,\delta\gamma}\frac{\delta W}{\delta\beta} + \frac{\delta^2 v}{\delta\gamma^2}\frac{\delta W}{\delta\gamma}\right).
\end{aligned}\right\} \tag{H$'$}$$

The equation of the caustic surfaces in x, y, z, may also be deduced from the characteristic function W, by eliminating α, β, γ, between the equations (K) and the following

$$W'' = 0: \tag{I$'$}$$

or from the function V, by simply putting

$$\frac{1}{V''} = 0. \tag{K$'$}$$

9. The formulæ of the preceding number determine by differentiations and eliminations alone, the equation of the two caustic surfaces; but when it is required to determine also the two series of caustic curves contained on these two surfaces, or the two series of developable pencils composed by the tangents to these curves, we must then have recourse to integration. The differential equation in x, y, z, which determines the developable pencils, may be found by eliminating ρ between the formulæ marked (B$'$), and may be put under any one of the three following forms:

$$\left.\begin{aligned}
\delta'\frac{\delta v}{\delta\alpha}\,.\,\delta\frac{\delta V}{\delta y} &= \delta'\frac{\delta v}{\delta\beta}\,.\,\delta\frac{\delta V}{\delta x},\\
\delta'\frac{\delta v}{\delta\beta}\,.\,\delta\frac{\delta V}{\delta z} &= \delta'\frac{\delta v}{\delta\gamma}\,.\,\delta\frac{\delta V}{\delta y},\\
\delta'\frac{\delta v}{\delta\gamma}\,.\,\delta\frac{\delta V}{\delta x} &= \delta'\frac{\delta v}{\delta\alpha}\,.\,\delta\frac{\delta V}{\delta z},
\end{aligned}\right\} \tag{L$'$}$$

in which α, β, γ, are considered as given function of x, y, z, deduced from the equations (C). The developable pencils having been thus determined, by integrating the equations (L$'$), the caustic curves will be known, because they are the *arêtes de rebroussement* of those pencils; the caustic curves may also be found by the condition of being contained at once on the developable pencils and on the caustic surfaces; or, finally, we may find the differential equations of those curves in x'', y'', z'', without reference to the developable pencils, by combining with the formulæ (F$'$) the differential relation between α, β, γ, which results from the equations (B$'$) and admits of being put under any one of the three following forms:

$$\left.\begin{aligned}
\delta'\frac{\delta v}{\delta\alpha}\,.\,\delta\frac{\delta v}{\delta\beta} &= \delta'\frac{\delta v}{\delta\beta}\,.\,\delta\frac{\delta v}{\delta\alpha},\\
\delta'\frac{\delta v}{\delta\beta}\,.\,\delta\frac{\delta v}{\delta\gamma} &= \delta'\frac{\delta v}{\delta\gamma}\,.\,\delta\frac{\delta v}{\delta\beta},\\
\delta'\frac{\delta v}{\delta\gamma}\,.\,\delta\frac{\delta v}{\delta\alpha} &= \delta'\frac{\delta v}{\delta\alpha}\,.\,\delta\frac{\delta v}{\delta\gamma};
\end{aligned}\right\} \tag{M$'$}$$

$$\delta' \frac{\delta v}{\delta \alpha}, \quad \delta' \frac{\delta v}{\delta \beta}, \quad \delta' \frac{\delta v}{\delta \gamma},$$

being changed to their expressions (P), or rather to the equivalent expressions,

$$\left.\begin{aligned}
\delta' \frac{\delta v}{\delta \alpha} &= M_{\prime}\, \delta\alpha + M'_{\prime}\, \delta\beta + P'_{\prime}\, \delta\gamma + (\alpha x + \beta y + \gamma z)\, \delta \frac{\delta v}{\delta \alpha}, \\
\delta' \frac{\delta v}{\delta \beta} &= M'_{\prime}\, \delta\alpha + N_{\prime}\, \delta\beta + N'_{\prime}\, \delta\gamma + (\alpha x + \beta y + \gamma z)\, \delta \frac{\delta v}{\delta \beta}, \\
\delta' \frac{\delta v}{\delta \gamma} &= P'_{\prime}\, \delta\alpha + N'_{\prime}\, \delta\beta + P_{\prime}\, \delta\gamma + (\alpha x + \beta y + \gamma z)\, \delta \frac{\delta v}{\delta \gamma},
\end{aligned}\right\} \quad (\mathrm{N}')$$

from which $\alpha x + \beta y + \gamma z$ will disappear, when substituted in the equations (M′), and in which

$$\left.\begin{aligned}
M_{\prime}\, \delta\alpha + M'_{\prime}\, \delta\beta + P'_{\prime}\, \delta\gamma &= \delta \frac{\delta W}{\delta \alpha} - \left(x_{\prime}\, \delta \frac{\delta^2 v}{\delta \alpha^2} + y_{\prime}\, \delta \frac{\delta^2 v}{\delta \alpha\, \delta \beta} + z_{\prime}\, \delta \frac{\delta^2 v}{\delta \alpha\, \delta \gamma} \right), \\
M'_{\prime}\, \delta\alpha + N_{\prime}\, \delta\beta + N'_{\prime}\, \delta\gamma &= \delta \frac{\delta W}{\delta \beta} - \left(x_{\prime}\, \delta \frac{\delta^2 v}{\delta \alpha\, \delta \beta} + y_{\prime}\, \delta \frac{\delta^2 v}{\delta \beta^2} + z_{\prime}\, \delta \frac{\delta^2 v}{\delta \beta\, \delta \gamma} \right), \\
P'_{\prime}\, \delta\alpha + N'_{\prime}\, \delta\beta + P_{\prime}\, \delta\gamma &= \delta \frac{\delta W}{\delta \gamma} - \left(x_{\prime}\, \delta \frac{\delta^2 v}{\delta \alpha\, \delta \gamma} + y_{\prime}\, \delta \frac{\delta^2 v}{\delta \beta\, \delta \gamma} + z_{\prime}\, \delta \frac{\delta^2 v}{\delta \gamma^2} \right).
\end{aligned}\right\} \quad (\mathrm{O}')$$

10. A remarkable transformation of the equations (B′), which determine, as we have seen, the developable pencils, and the caustic curves and surfaces, may be obtained in the following manner. We have by (P),

$$\delta' \frac{\delta v}{\delta \alpha} = \delta \frac{\delta W}{\delta \alpha} - \left(x\, \delta \frac{\delta^2 v}{\delta \alpha^2} + y\, \delta \frac{\delta^2 v}{\delta \alpha\, \delta \beta} + z\, \delta \frac{\delta^2 v}{\delta \alpha\, \delta \gamma} \right),$$

which gives

$$\delta' \frac{\delta v}{\delta \alpha} - \rho\, \delta \frac{\delta v}{\delta \alpha} = \delta \frac{\delta W}{\delta \alpha} - \left(x''\, \delta \frac{\delta^2 v}{\delta \alpha^2} + y''\, \delta \frac{\delta^2 v}{\delta \alpha\, \delta \beta} + z''\, \delta \frac{\delta^2 v}{\delta \alpha\, \delta \gamma} \right),$$

when we substitute for x, y, z, their expressions $x'' + \alpha\rho$, $y'' + \beta\rho$, $z'' + \gamma\rho$, and attend to the relations (G). And by similar substitutions in the expressions for

$$\delta' \frac{\delta v}{\delta \beta}, \quad \text{and} \quad \delta' \frac{\delta v}{\delta \gamma},$$

the equations (B′) become,

$$\left.\begin{aligned}
\delta \frac{\delta W}{\delta \alpha} &= x''\, \delta \frac{\delta^2 v}{\delta \alpha^2} + y''\, \delta \frac{\delta^2 v}{\delta \alpha\, \delta \beta} + z''\, \delta \frac{\delta^2 v}{\delta \alpha\, \delta \gamma}, \\
\delta \frac{\delta W}{\delta \beta} &= x''\, \delta \frac{\delta^2 v}{\delta \alpha\, \delta \beta} + y''\, \delta \frac{\delta^2 v}{\delta \beta^2} + z''\, \delta \frac{\delta^2 v}{\delta \beta\, \delta \gamma}, \\
\delta \frac{\delta W}{\delta \gamma} &= x''\, \delta \frac{\delta^2 v}{\delta \alpha\, \delta \gamma} + y''\, \delta \frac{\delta^2 v}{\delta \beta\, \delta \gamma} + z''\, \delta \frac{\delta^2 v}{\delta \gamma^2}.
\end{aligned}\right\} \quad (\mathrm{P}')$$

Now, if we conceive another system of rays, composed of the same kind of light, and contained in the same medium, but all converging to or diverging from the one point, x'', y'', z'', and represent by W', the characteristic function, which in this new system, corresponds to W in the old, we shall have

$$\left.\begin{aligned}
\frac{\delta W'}{\delta\alpha} &= x''\frac{\delta^2 v}{\delta\alpha^2} + y''\frac{\delta^2 v}{\delta\alpha\,\delta\beta} + z''\frac{\delta^2 v}{\delta\alpha\,\delta\gamma},\\
\frac{\delta W'}{\delta\beta} &= x''\frac{\delta^2 v}{\delta\alpha\,\delta\beta} + y''\frac{\delta^2 v}{\delta\beta^2} + z''\frac{\delta^2 v}{\delta\beta\,\delta\gamma},\\
\frac{\delta W'}{\delta\gamma} &= x''\frac{\delta^2 v}{\delta\alpha\,\delta\gamma} + y''\frac{\delta^2 v}{\delta\beta\,\delta\gamma} + z''\frac{\delta^2 v}{\delta\gamma^2},\\
\delta\frac{\delta W'}{\delta\alpha} &= x''\,\delta\frac{\delta^2 v}{\delta\alpha^2} + y''\,\delta\frac{\delta^2 v}{\delta\alpha\,\delta\beta} + z''\,\delta\frac{\delta^2 v}{\delta\alpha\,\delta\gamma},\\
\delta\frac{\delta W'}{\delta\beta} &= x''\,\delta\frac{\delta^2 v}{\delta\alpha\,\delta\beta} + y''\,\delta\frac{\delta^2 v}{\delta\beta^2} + z''\,\delta\frac{\delta^2 v}{\delta\beta\,\delta\gamma},\\
\delta\frac{\delta W'}{\delta\gamma} &= x''\,\delta\frac{\delta^2 v}{\delta\alpha\,\delta\gamma} + y''\,\delta\frac{\delta^2 v}{\delta\beta\,\delta\gamma} + z''\,\delta\frac{\delta^2 v}{\delta\gamma^2};
\end{aligned}\right\} \qquad (\mathrm{Q}')$$

the equations which determine the developable pencils, and the caustic curves and surfaces, may therefore be thus written:

$$\delta\frac{\delta W}{\delta\alpha} = \delta\frac{\delta W'}{\delta\alpha};\quad \delta\frac{\delta W}{\delta\beta} = \delta\frac{\delta W'}{\delta\beta};\quad \delta\frac{\delta W}{\delta\gamma} = \delta\frac{\delta W'}{\delta\gamma}. \qquad (\mathrm{R}')$$

On Osculating Focal Systems.

11. The equations which we have thus obtained, as transformations of the formulæ (B′), are not only remarkable in an analytic view, but contain an interesting geometrical property of the caustic surfaces. To explain this property, it is necessary to introduce the consideration of osculating systems of rays. Let us therefore conceive a system, placed in the same medium, and composed of the same kind of light, as that given system of rays which has W for its characteristic function, but converging to or diverging from some one point X, Y, Z; and let us denote by W', the corresponding characteristic function of this new system, which becomes equal to the W' of the preceding number, when the point X, Y, Z, coincides with the point x'', y'', z''; then the general expression for this function W' is

$$W' = X\frac{\delta v}{\delta\alpha} + Y\frac{\delta v}{\delta\beta} + Z\frac{\delta v}{\delta\gamma} + C, \qquad (\mathrm{S}')$$

C being an arbitrary constant; and the system which thus has W' for its characteristic function, we shall call a *focal system.* The four arbitrary quantities, X, Y, Z, C, which enter into the general expression (S′) for W', may be determined by the condition that for some given ray of the given system, that is, for some given values of α, β, γ, certain of the first terms of the development of W', according to the positive powers of the variations of α, β, γ, may be equal to the corresponding terms in the development of the given function W; and when the form of W' has been particularized by this condition, we shall call the corresponding system of rays an *osculating focal system.* Now, if we suppose α, β, γ, to be changed into $\alpha + \delta\alpha$, $\beta + \delta\beta$, $\gamma + \delta\gamma$, we may express the altered values of W and W' by means of the following developments:

$$W + \delta W + \frac{1}{2}\delta^2 W + \frac{1}{2\,.\,3}\delta^3 W + \&\text{c.},$$

$$W' + \delta W' + \frac{1}{2}\delta^2 W' + \frac{1}{2\,.\,3}\delta^3 W' + \&\text{c.},$$

in which

$$\begin{aligned}
\delta W &= \frac{\delta W}{\delta\alpha}\,\delta\alpha + \frac{\delta W}{\delta\beta}\,\delta\beta + \frac{\delta W}{\delta\gamma}\,\delta\gamma, \\
\delta^2 W &= \frac{\delta^2 W}{\delta\alpha^2}\,\delta\alpha^2 + \frac{\delta^2 W}{\delta\beta^2}\,\delta\beta^2 + \frac{\delta^2 W}{\delta\gamma^2}\,\delta\gamma^2 + 2\frac{\delta^2 W}{\delta\alpha\,\delta\beta}\,\delta\alpha\,\delta\beta + 2\frac{\delta^2 W}{\delta\beta\,\delta\gamma}\,\delta\beta\,\delta\gamma + 2\frac{\delta^2 W}{\delta\gamma\,\delta\alpha}\,\delta\gamma\,\delta\alpha, \\
&\&\text{c.}
\end{aligned}$$

The equations

$$W' = W, \quad \delta W' = \delta W, \tag{T$'$}$$

will be satisfied independently of the ratios of the variations $\delta\alpha$, $\delta\beta$, $\delta\gamma$, if we take the point X, Y, Z, any where upon the given ray, and suppose,

$$C = W - \left(X\,\frac{\delta v}{\delta\alpha} + Y\,\frac{\delta v}{\delta\beta} + Z\,\frac{\delta v}{\delta\gamma} \right).$$

There remains therefore one arbitrary constant of the focal system to be determined, and this is to be done by equating the next terms of the developments, that is by putting

$$\delta^2 W' = \delta^2 W, \tag{U$'$}$$

and assigning some limiting ratios to the variations $\delta\alpha$, $\delta\beta$, $\delta\gamma$, consistent with the differential equation

$$\alpha\,\delta\alpha + \beta\,\delta\beta + \gamma\,\delta\gamma = 0,$$

which results from $\alpha^2 + \beta^2 + \gamma^2 = 1$. And, from the nature of the functions W, W', the equation (U$'$) may be put under the following form:

$$\begin{aligned}
0 = {} & \left(\frac{\delta^2 W'}{\delta\alpha^2} - \frac{\delta^2 W}{\delta\alpha^2} \right) \left(\delta\alpha - \frac{\alpha}{\gamma}\,\delta\gamma \right)^2 + 2\left(\frac{\delta^2 W'}{\delta\alpha\,\delta\beta} - \frac{\delta^2 W}{\delta\alpha\,\delta\beta} \right) \left(\delta\alpha - \frac{\alpha}{\gamma}\,\delta\gamma \right) \left(\delta\beta - \frac{\beta}{\gamma}\,\delta\gamma \right) \\
& + \left(\frac{\delta^2 W'}{\delta\beta^2} - \frac{\delta^2 W}{\delta\beta^2} \right) \left(\delta\beta - \frac{\beta}{\gamma}\,\delta\gamma \right)^2;
\end{aligned} \tag{V$'$}$$

which shews that there are in general an infinite number of osculating focal systems corresponding to any given ray, that is, an infinite number of different values for the arbitrary parameter which enters into the expressions of

$$\frac{\delta^2 W'}{\delta\alpha^2}, \quad \frac{\delta^2 W'}{\delta\alpha\,\delta\beta}, \quad \frac{\delta^2 W'}{\delta\beta^2},$$

according to the infinite variety of values that we may assign to the ratio

$$\frac{\gamma\,\delta\beta - \beta\,\delta\gamma}{\gamma\,\delta\alpha - \alpha\,\delta\gamma};$$

but that the values of this arbitrary parameter, which do not change for an infinitely small alteration in the ratio on which they depend, are determined by the following equations:

$$\left.\begin{aligned} 0 &= \left(\frac{\delta^2 W'}{\delta\alpha^2} - \frac{\delta^2 W}{\delta\alpha^2}\right)\left(\delta\alpha - \frac{\alpha}{\gamma}\,\delta\gamma\right) + \left(\frac{\delta^2 W'}{\delta\alpha\,\delta\beta} - \frac{\delta^2 W}{\delta\alpha\,\delta\beta}\right)\left(\delta\beta - \frac{\beta}{\gamma}\,\delta\gamma\right), \\ 0 &= \left(\frac{\delta^2 W'}{\delta\alpha\,\delta\beta} - \frac{\delta^2 W}{\delta\alpha\,\delta\beta}\right)\left(\delta\alpha - \frac{\alpha}{\gamma}\,\delta\gamma\right) + \left(\frac{\delta^2 W'}{\delta\beta^2} - \frac{\delta^2 W}{\delta\beta^2}\right)\left(\delta\beta - \frac{\beta}{\gamma}\,\delta\gamma\right); \end{aligned}\right\} \tag{W$'$}$$

which give, by elimination,

$$\left(\frac{\delta^2 W'}{\delta\alpha^2} - \frac{\delta^2 W}{\delta\alpha^2}\right)\left(\frac{\delta^2 W'}{\delta\beta^2} - \frac{\delta^2 W}{\delta\beta^2}\right) = \left(\frac{\delta^2 W'}{\delta\alpha\,\delta\beta} - \frac{\delta^2 W}{\delta\alpha\,\delta\beta}\right)^2. \tag{X$'$}$$

The systems that correspond to these extreme values of the arbitrary parameter, we shall call the *extreme osculating focal systems*; and since, by the nature of the functions W, W', the equations (W$'$) are equivalent to the formulæ (R$'$), the foci of these extreme osculating systems are contained upon the caustic surfaces: and the ratios of $\delta\alpha$, $\delta\beta$, $\delta\gamma$, in these extreme systems, are the same as in the developable pencils.

12. Let us consider the law of the variation of the focus of the osculating system, between its limiting positions. This law is analytically expressed by the formula (U$'$); in which we may geometrically interpret $\delta\alpha$, $\delta\beta$, $\delta\gamma$, by considering these infinitely small variations of α, β, γ, as arising in the passage from the given ray to an infinitely near ray of the system. The plane which passes through the given ray, and is parallel to the infinitely near ray, may be called the *plane of osculation*; since, if it be known, we shall know the ratios of $\delta\alpha$, $\delta\beta$, $\delta\gamma$, and can determine, by the formula (U$'$), the position of the focus of the osculating system. To simplify this determination, let us put

$$X = x_{\prime} + \alpha R, \quad Y = y_{\prime} + \beta R, \quad Z = z_{\prime} + \gamma R, \tag{Y$'$}$$

X, Y, Z, being the coordinates of the focus, and $x_{\prime}$, $y_{\prime}$, $z_{\prime}$, having the same meanings as in the eighth number; the formula (U$'$) then becomes, by the nature of W', and by the relations (G),

$$R\,\delta^2 v + \delta^2 W = x_{\prime}\,\delta^2\frac{\delta v}{\delta\alpha} + y_{\prime}\,\delta^2\frac{\delta v}{\delta\beta} + z_{\prime}\,\delta^2\frac{\delta v}{\delta\gamma}, \tag{Z$'$}$$

$\delta^2 v$ denoting

$$\delta\alpha\,\delta\frac{\delta v}{\delta\alpha} + \delta\beta\,\delta\frac{\delta v}{\delta\beta} + \delta\gamma\,\delta\frac{\delta v}{\delta\gamma}.$$

The second member of this equation (Z′) vanishes when the ray passes through the origin; and if we suppose the ray to coincide with the axis of z, we shall have also $\delta\gamma = 0$, and the equation will become,

$$0 = \left(R\frac{\delta^2 v}{\delta\alpha^2} + \frac{\delta^2 W}{\delta\alpha^2}\right)\delta\alpha^2 + 2\left(R\frac{\delta^2 v}{\delta\alpha\,\delta\beta} + \frac{\delta^2 W}{\delta\alpha\,\delta\beta}\right)\delta\alpha\,\delta\beta + \left(R\frac{\delta^2 v}{\delta\beta^2} + \frac{\delta^2 W}{\delta\beta^2}\right)\delta\beta^2, \qquad (\text{A}'')$$

which expresses the dependence of the parameter R, on the ratio of $\delta\beta$ to $\delta\alpha$; R being now the distance from the origin, upon the ray, to the focus of the osculating system; and the ratio $\dfrac{\delta\beta}{\delta\alpha}$ being the tangent of the angle ϕ, comprised between the plane of xz and the plane having for equation,

$$\frac{y}{x} = \frac{\delta\beta}{\delta\alpha} = \tan.\phi, \qquad (\text{B}'')$$

that is the plane of osculation. This plane is tangent to one of the developable pencils, when the distance R attains either of its extreme values, corresponding to the two points where the ray touches the caustic surfaces, and is determined by the equation,

$$\left(R\frac{\delta^2 v}{\delta\alpha^2} + \frac{\delta^2 W}{\delta\alpha^2}\right)\left(R\frac{\delta^2 v}{\delta\beta^2} + \frac{\delta^2 W}{\delta\beta^2}\right) = \left(R\frac{\delta^2 v}{\delta\alpha\,\delta\beta} + \frac{\delta^2 W}{\delta\alpha\,\delta\beta}\right)^2, \qquad (\text{C}'')$$

which results by elimination from the two following:

$$\left.\begin{aligned} 0 &= \left(R\frac{\delta^2 v}{\delta\alpha^2} + \frac{\delta^2 W}{\delta\alpha^2}\right) + \left(R\frac{\delta^2 v}{\delta\alpha\,\delta\beta} + \frac{\delta^2 W}{\delta\alpha\,\delta\beta}\right)\tan.\phi, \\ 0 &= \left(R\frac{\delta^2 v}{\delta\alpha\,\delta\beta} + \frac{\delta^2 W}{\delta\alpha\,\delta\beta}\right) + \left(R\frac{\delta^2 v}{\delta\beta^2} + \frac{\delta^2 W}{\delta\beta^2}\right)\tan.\phi. \end{aligned}\right\} \qquad (\text{D}'')$$

Let R_1, R_2, be the two values of R, determined by the formula (C″), and ϕ_1, ϕ_2, the two corresponding values of the angle ϕ, which may be deduced from the following equation:

$$\begin{aligned} &\left(\frac{\delta^2 v}{\delta\alpha^2} + \frac{\delta^2 v}{\delta\alpha\,\delta\beta}\tan.\phi\right)\left(\frac{\delta^2 W}{\delta\alpha\,\delta\beta} + \frac{\delta^2 W}{\delta\beta^2}\tan.\phi\right) \\ = &\left(\frac{\delta^2 v}{\delta\alpha\,\delta\beta} + \frac{\delta^2 v}{\delta\beta^2}\tan.\phi\right)\left(\frac{\delta^2 W}{\delta\alpha^2} + \frac{\delta^2 W}{\delta\alpha\,\delta\beta}\tan.\phi\right); \end{aligned} \qquad (\text{E}'')$$

then the general relation (A″) between R and ϕ, may be put under the following form:

$$\frac{R - R_1}{R_2 - R} = \zeta\left(\frac{\sin.(\phi - \phi_1)}{\sin.(\phi_2 - \phi)}\right)^2, \qquad (\text{F}'')$$

ζ being a coefficient which is independent of R and ϕ, and is positive or negative according as the quantity

$$\frac{\delta^2 v}{\delta\alpha^2}\frac{\delta^2 v}{\delta\beta^2} - \left(\frac{\delta^2 v}{\delta\alpha\,\delta\beta}\right)^2$$

is positive or negative. This latter quantity is the same with that which we have before denoted by v'', because the remaining parts of the general expressions for v'', namely

$$v'' = \frac{\delta^2 v}{\delta\alpha^2}\frac{\delta^2 v}{\delta\beta^2} - \left(\frac{\delta^2 v}{\delta\alpha\,\delta\beta}\right)^2 + \frac{\delta^2 v}{\delta\beta^2}\frac{\delta^2 v}{\delta\gamma^2} - \left(\frac{\delta^2 v}{\delta\beta\,\delta\gamma}\right)^2 + \frac{\delta^2 v}{\delta\gamma^2}\frac{\delta^2 v}{\delta\alpha^2} - \left(\frac{\delta^2 v}{\delta\gamma\,\delta\alpha}\right)^2,$$

vanish when $\alpha = 0$, $\beta = 0$. If therefore v'' be positive, and if we denote by R_2 the greater of the two values R_1, R_2, that is the one nearer to positive infinity, we shall have by (F$''$), for all other values of R,

$$R > R_1, \quad R < R_2, \quad (v'' > 0); \tag{G$''$}$$

so that in this case the foci of the osculating systems are all ranged upon that finite portion of the ray which lies between the caustic surfaces. If, on the contrary, v'' is negative, then the two differences $R - R_1$ and $R - R_2$ are both positive or both negative, so that

$$\frac{R - R_1}{R - R_2} > 0, \quad (v'' < 0); \tag{H$''$}$$

in this case, therefore, the foci of the osculating systems are all contained upon the remainder of the ray, that is upon the two indefinite portions which lie outside the former interval. And in each case, the distances of the focus of any osculating system from the two points in which the ray touches the two caustic surfaces, are proportional to the squares of the sines of the angles which the plane of osculation makes with the two tangent planes to the developable pencils. In the foregoing investigations we have supposed that W, and its analogous function W', which we consider for symmetry as homogeneous, are put under the form of functions of the dimension zero; a supposition which permits us to adopt the expressions (K) for the partial differentials

$$\frac{\delta W}{\delta\alpha}, \quad \frac{\delta W}{\delta\beta}, \quad \frac{\delta W}{\delta\gamma},$$

instead of the less simple and more general expressions given in the fourth number: but if we had assigned any other value to the dimension i, in those more general expressions, we should have deduced the same results respecting the law of osculation.

13. The function v'', the sign of which distinguishes between the two preceding cases of osculation, has this remarkable property, that it is independent of the direction of the coordinate axes; in such a manner that if α, β, γ, be, as before, the cosines of the angles which the ray makes with three given rectangular axes, and if we denote by α', β', γ', the new values which these cosines acquire when we refer the ray to three new rectangular axes, we shall have

$$\frac{\delta^2 v}{\delta\alpha^2}\frac{\delta^2 v}{\delta\beta^2} - \left(\frac{\delta^2 v}{\delta\alpha\,\delta\beta}\right)^2 + \frac{\delta^2 v}{\delta\beta^2}\frac{\delta^2 v}{\delta\gamma^2} - \left(\frac{\delta^2 v}{\delta\beta\,\delta\gamma}\right)^2 + \frac{\delta^2 v}{\delta\gamma^2}\frac{\delta^2 v}{\delta\alpha^2} - \left(\frac{\delta^2 v}{\delta\gamma\,\delta\alpha}\right)^2$$

$$= \frac{\delta^2 v}{\delta\alpha'^2}\frac{\delta^2 v}{\delta\beta'^2} - \left(\frac{\delta^2 v}{\delta\alpha'\,\delta\beta'}\right)^2 + \frac{\delta^2 v}{\delta\beta'^2}\frac{\delta^2 v}{\delta\gamma'^2} - \left(\frac{\delta^2 v}{\delta\beta'\,\delta\gamma'}\right)^2 + \frac{\delta^2 v}{\delta\gamma'^2}\frac{\delta^2 v}{\delta\alpha'^2} - \left(\frac{\delta^2 v}{\delta\gamma'\,\delta\alpha'}\right)^2 : \tag{I$''$}$$

v being, in the first member, a homogeneous function of α, β, γ, and, in the second member of α', β', γ', of the first dimension. To demonstrate this theorem, let us observe that by the known formulæ for the transformation of coordinates, we may put

$$\left.\begin{array}{ll} \alpha = \alpha' A + \beta' B + \gamma' C, & \alpha' = \alpha A + \beta A' + \gamma A'', \\ \beta = \alpha' A' + \beta' B' + \gamma' C', & \beta' = \alpha B + \beta B' + \gamma B'', \\ \gamma = \alpha' A'' + \beta' B'' + \gamma' C'', & \gamma' = \alpha C + \beta C' + \gamma C''; \end{array}\right\} \tag{K$''$}$$

A, B, C, A', B', C', A'', B'', C'', being constant quantities of which only three are arbitrary, and which satisfy the following conditions:

$$\left.\begin{array}{rr} A^2 + B^2 + C^2 = 1, & A^2 + A'^2 + A''^2 = 1, \\ A'^2 + B'^2 + C'^2 = 1, & B^2 + B'^2 + B''^2 = 1, \\ A''^2 + B''^2 + C''^2 = 1, & C^2 + C'^2 + C''^2 = 1, \\ AA' + BB' + CC' = 0, & AB + A'B' + A''B'' = 0, \\ A'A'' + B'B'' + C'C'' = 0, & BC + B'C' + B''C'' = 0, \\ A''A + B''B + C''C = 0, & CA + C'A' + C''A'' = 0. \end{array}\right\} \tag{L$''$}$$

This being laid down, we have, by (K$''$), and by the nature of partial differentials,

$$\begin{aligned} \frac{\delta v}{\delta\alpha'} &= A\frac{\delta v}{\delta\alpha} + A'\frac{\delta v}{\delta\beta} + A''\frac{\delta v}{\delta\gamma}, \\ \frac{\delta v}{\delta\beta'} &= B\frac{\delta v}{\delta\alpha} + B'\frac{\delta v}{\delta\beta} + B''\frac{\delta v}{\delta\gamma}, \\ \frac{\delta v}{\delta\gamma'} &= C\frac{\delta v}{\delta\alpha} + C'\frac{\delta v}{\delta\beta} + C''\frac{\delta v}{\delta\gamma}. \end{aligned}$$

and, continuing the differentiations,

$$\begin{aligned} \frac{\delta^2 v}{\delta\alpha'^2} &= A^2\frac{\delta^2 v}{\delta\alpha^2} + A'^2\frac{\delta^2 v}{\delta\beta^2} + A''^2\frac{\delta^2 v}{\delta\gamma^2} + 2AA'\frac{\delta^2 v}{\delta\alpha\,\delta\beta} + 2A'A''\frac{\delta^2 v}{\delta\beta\,\delta\gamma} + 2A''A\frac{\delta^2 v}{\delta\gamma\,\delta\alpha}, \\ \frac{\delta^2 v}{\delta\beta'^2} &= B^2\frac{\delta^2 v}{\delta\alpha^2} + B'^2\frac{\delta^2 v}{\delta\beta^2} + B''^2\frac{\delta^2 v}{\delta\gamma^2} + 2BB'\frac{\delta^2 v}{\delta\alpha\,\delta\beta} + 2B'B''\frac{\delta^2 v}{\delta\beta\,\delta\gamma} + 2B''B\frac{\delta^2 v}{\delta\gamma\,\delta\alpha}, \\ \frac{\delta^2 v}{\delta\gamma'^2} &= C^2\frac{\delta^2 v}{\delta\alpha^2} + C'^2\frac{\delta^2 v}{\delta\beta^2} + C''^2\frac{\delta^2 v}{\delta\gamma^2} + 2CC'\frac{\delta^2 v}{\delta\alpha\,\delta\beta} + 2C'C''\frac{\delta^2 v}{\delta\beta\,\delta\gamma} + 2C''C\frac{\delta^2 v}{\delta\gamma\,\delta\alpha}, \end{aligned}$$

$$\frac{\delta^2 v}{\delta\alpha'\,\delta\beta'} = A\left(B\frac{\delta^2 v}{\delta\alpha^2} + B'\frac{\delta^2 v}{\delta\alpha\,\delta\beta} + B''\frac{\delta^2 v}{\delta\alpha\,\delta\gamma}\right) + A'\left(B\frac{\delta^2 v}{\delta\alpha\,\delta\beta} + B'\frac{\delta^2 v}{\delta\beta^2} + B''\frac{\delta^2 v}{\delta\beta\,\delta\gamma}\right)$$
$$+ A''\left(B\frac{\delta^2 v}{\delta\alpha\,\delta\gamma} + B'\frac{\delta^2 v}{\delta\beta\,\delta\gamma} + B''\frac{\delta^2 v}{\delta\gamma^2}\right),$$
$$\frac{\delta^2 v}{\delta\beta'\,\delta\gamma'} = B\left(C\frac{\delta^2 v}{\delta\alpha^2} + C'\frac{\delta^2 v}{\delta\alpha\,\delta\beta} + C''\frac{\delta^2 v}{\delta\alpha\,\delta\gamma}\right) + B'\left(C\frac{\delta^2 v}{\delta\alpha\,\delta\beta} + C'\frac{\delta^2 v}{\delta\beta^2} + C''\frac{\delta^2 v}{\delta\beta\,\delta\gamma}\right)$$
$$+ B''\left(C\frac{\delta^2 v}{\delta\alpha\,\delta\gamma} + C'\frac{\delta^2 v}{\delta\beta\,\delta\gamma} + C''\frac{\delta^2 v}{\delta\gamma^2}\right),$$
$$\frac{\delta^2 v}{\delta\gamma'\,\delta\alpha'} = C\left(A\frac{\delta^2 v}{\delta\alpha^2} + A'\frac{\delta^2 v}{\delta\alpha\,\delta\beta} + A''\frac{\delta^2 v}{\delta\alpha\,\delta\gamma}\right) + C'\left(A\frac{\delta^2 v}{\delta\alpha\,\delta\beta} + A'\frac{\delta^2 v}{\delta\beta^2} + A''\frac{\delta^2 v}{\delta\beta\,\delta\gamma}\right)$$
$$+ C''\left(A\frac{\delta^2 v}{\delta\alpha\,\delta\gamma} + A'\frac{\delta^2 v}{\delta\beta\,\delta\gamma} + A''\frac{\delta^2 v}{\delta\gamma^2}\right);$$

and substituting these values for

$$\frac{\delta^2 v}{\delta\alpha'^2}, \quad \frac{\delta^2 v}{\delta\beta'^2}, \quad \frac{\delta^2 v}{\delta\gamma'^2}, \quad \frac{\delta^2 v}{\delta\alpha'\,\delta\beta'}, \quad \frac{\delta^2 v}{\delta\beta'\,\delta\gamma'}, \quad \frac{\delta^2 v}{\delta\gamma'\,\delta\alpha'};$$

in the second member of (I″), and reducing by the relations (K″), (L″), and (G), we obtain the function in the first member. The function v'', which composes the first member of (I″), may therefore be obtained by assigning to the axes of coordinates, any arbitrary but rectangular directions, which may most facilitate the calculation. For example, when we are considering an extraordinary system of rays in a one-axed crystal, we may take the axis of the crystal for the axis of z, and then the function v will take the form

$$v = \sqrt{m^2\gamma^2 + n^2(\alpha^2 + \beta^2)}, \tag{M″}$$

the quantities m, n, being independent of α, β, γ; and we find by differentiation,

$$\left.\begin{array}{lll} v\dfrac{\delta v}{\delta\alpha} = n^2\alpha, & v\dfrac{\delta v}{\delta\beta} = n^2\beta, & v\dfrac{\delta v}{\delta\gamma} = m^2\gamma, \\ \dfrac{v^3}{n^2}\dfrac{\delta^2 v}{\delta\alpha^2} = m^2\gamma^2 + n^2\beta^2, & \dfrac{v^3}{n^2}\dfrac{\delta^2 v}{\delta\beta^2} = m^2\gamma^2 + n^2\alpha^2, & \dfrac{v^3}{n^2}\dfrac{\delta^2 v}{\delta\gamma^2} = m^2(\alpha^2 + \beta^2), \\ \dfrac{v^3}{n^2}\dfrac{\delta^2 v}{\delta\alpha\,\delta\beta} = -n^2\alpha\beta, & \dfrac{v^3}{n^2}\dfrac{\delta^2 v}{\delta\beta\,\delta\gamma} = -m^2\beta\gamma, & \dfrac{v^3}{n^2}\dfrac{\delta^2 v}{\delta\gamma\,\delta\alpha} = -m^2\gamma\alpha, \end{array}\right\} \tag{N″}$$

values which may be verified by the relations (G), and which give

$$v'' = \frac{m^2 n^4(\alpha^2 + \beta^2 + \gamma^2)}{v^4} = \frac{m^2 n^4}{v^4}: \tag{O″}$$

we may therefore conclude that whatever be the directions of the rectangular axes of coordinates in an extraordinary system of this kind, the function v'' is essentially positive, and is equal to the square of the constant m, multiplied by the fourth power of the constant n, and divided by the fourth power of v; v being the velocity of the extraordinary rays of some given colour, estimated on the hypothesis of molecular emission, and the constants m, n, being the values which v assumes when the ray becomes respectively parallel and perpendicular to the optical axis of the crystal. Hence it follows, that in extraordinary systems of this kind, the foci of the osculating systems, considered in the preceding number, are all comprised between the two points in which the given ray touches the two caustic surfaces. It is evident that this result extends to the case of ordinary systems of rays, to which the expressions (M″), (N″), for v, and for its partial differentials, may be adapted by making $n = m$, a change which gives, by (O″), $v'' = m^2$.

Principal Foci and Principal Rays.

14. Another important property of the function v'', is that when, by the nature of the light and of the medium, this function is essentially greater than zero, (which we have seen to be the case for all ordinary systems of rays, and for the extraordinary systems produced by one-axed crystals,) the intersection of the two caustic surfaces reduces itself in general to a finite number of isolated points. To prove this theorem, let us resume the formulæ of the twelfth number, and let us suppose that the ray which coincides with the axis of z, passes through a point of intersection of the caustic surfaces, so that the two roots of the quadratic (C″) are equal; then the two values of $\tan.\phi$, deduced from the quadratic (E″), will be equal also; and if we put this quadratic under the form

$$E(\tan.\phi)^2 - E'\tan.\phi + E'' = 0, \tag{P''}$$

in which

$$E = \frac{\delta^2 v}{\delta\alpha\,\delta\beta}\frac{\delta^2 W}{\delta\beta^2} - \frac{\delta^2 v}{\delta\beta^2}\frac{\delta^2 W}{\delta\alpha\,\delta\beta},$$
$$E' = \frac{\delta^2 v}{\delta\beta^2}\frac{\delta^2 W}{\delta\alpha^2} - \frac{\delta^2 v}{\delta\alpha^2}\frac{\delta^2 W}{\delta\beta^2},$$
$$E'' = \frac{\delta^2 v}{\delta\alpha^2}\frac{\delta^2 W}{\delta\alpha\,\delta\beta} - \frac{\delta^2 v}{\delta\alpha\,\delta\beta}\frac{\delta^2 W}{\delta\alpha^2},$$

we must have

$$E'^2 - 4EE'' = 0. \tag{Q''}$$

Now the coefficients E, E', E'', are connected by the following relation:

$$E\frac{\delta^2 v}{\delta\alpha^2} + E'\frac{\delta^2 v}{\delta\alpha\,\delta\beta} + E''\frac{\delta^2 v}{\delta\beta^2} = 0; \tag{R''}$$

and it results from this relation, that if

$$\frac{\delta^2 v}{\delta\alpha^2}\frac{\delta^2 v}{\delta\beta^2} - \left(\frac{\delta^2 v}{\delta\alpha\,\delta\beta}\right)^2 > 0,$$

the condition (Q″) cannot be satisfied without supposing separately

$$E = 0, \quad E' = 0, \quad E'' = 0. \tag{S''}$$

We may therefore put

$$\frac{\delta^2 W}{\delta\alpha^2} = \mu\frac{\delta^2 v}{\delta\alpha^2}, \quad \frac{\delta^2 W}{\delta\alpha\,\delta\beta} = \mu\frac{\delta^2 v}{\delta\alpha\,\delta\beta}, \quad \frac{\delta^2 W}{\delta\beta^2} = \mu\frac{\delta^2 v}{\delta\beta^2},$$

μ being a quantity which can be determined by substituting these values in the quadratic (C$''$); for this substitution gives,

$$v''(R+\mu)^2 = 0, \quad \mu = -R,$$

R being the common value of the two equal roots. Hence it follows, that when R is made equal to this value in the equation (A$''$) for the focus of an osculating system, that is, when we place this focus at the intersection of the caustic surfaces, the coefficients of $\delta\alpha^2$, $2\,\delta\alpha\,\delta\beta$, $\delta\beta^2$, namely,

$$R\frac{\delta^2 v}{\delta\alpha^2} + \frac{\delta^2 W}{\delta\alpha^2}, \quad R\frac{\delta^2 v}{\delta\alpha\,\delta\beta} + \frac{\delta^2 W}{\delta\alpha\,\delta\beta}, \quad R\frac{\delta^2 v}{\delta\beta^2} + \frac{\delta^2 W}{\delta\beta^2},$$

become separately $= 0$; and it is easy to prove that in like manner the coefficients of

$$\left(\delta\alpha - \frac{\alpha}{\gamma}\,\delta\gamma\right)^2, \quad 2\left(\delta\alpha - \frac{\alpha}{\gamma}\,\delta\gamma\right)\left(\delta\beta - \frac{\beta}{\gamma}\,\delta\gamma\right), \quad \left(\delta\beta - \frac{\beta}{\gamma}\,\delta\gamma\right)^2,$$

must separately vanish, in the more general equation (V$'$) of the eleventh number; we have therefore generally, for the intersection of the caustic surfaces, when the function v'' is essentially > 0, the following equations:

$$\left.\begin{aligned}
\frac{\delta^2 W'}{\delta\alpha^2} &= \frac{\delta^2 W}{\delta\alpha^2}, & \frac{\delta^2 W'}{\delta\alpha\,\delta\beta} &= \frac{\delta^2 W}{\delta\alpha\,\delta\beta}, & \frac{\delta^2 W'}{\delta\beta^2} &= \frac{\delta^2 W}{\delta\beta^2},\\
\frac{\delta^2 W'}{\delta\alpha\,\delta\gamma} &= \frac{\delta^2 W}{\delta\alpha\,\delta\gamma}, & \frac{\delta^2 W'}{\delta\beta\,\delta\gamma} &= \frac{\delta^2 W}{\delta\beta\,\delta\gamma}, & \frac{\delta^2 W'}{\delta\gamma^2} &= \frac{\delta^2 W}{\delta\gamma^2},
\end{aligned}\right\} \qquad (\mathrm{T}'')$$

of which the three latter result from the three former. These six equations, which are all expressed by the one formula (U$'$) or (Z$'$), provided that we consider $\delta\alpha$, $\delta\beta$, $\delta\gamma$, as independent, will give in general a finite number of real or imaginary values for α, β, γ, R, and thus will determine a finite number of isolated points, as the intersection of the caustic surfaces. We shall call these points the *Principal Foci*; and the rays to which they belong, we shall call the *Principal Rays* of the system. In general, whether v'' be greater than or less than zero, we may employ the equations (T$''$) to determine a finite number of isolated points and rays, to which we shall give the same denominations. It results from the equations by which these points and rays are determined, that if the focus of an osculating system be placed at a principal focus of a given system, the osculation of the second order will be most complete, since it will be independent of the direction of the plane of osculation (B$''$); the three first terms of the two developments in the eleventh number, namely,

$$W + \delta W + \tfrac{1}{2}\delta^2 W, \quad W' + \delta W' + \tfrac{1}{2}\delta^2 W',$$

becoming equal, independently of the ratios of $\delta\alpha$, $\delta\beta$, $\delta\gamma$. The principal foci of an optical system possess many other remarkable properties, some of which we shall indicate in the course of the present supplement.

15. To develope one of the properties of the principal foci and principal rays of an optical system, we must introduce the consideration of osculating spheroids, and surfaces of constant action. The characteristic function V, the mode of dependence of which upon the coordinates x, y, z, distinguishes any one system of rays from any other, having the same kind of light and contained in the same medium, is equal, as we have seen, to the definite integral $\int v\,ds$, that is to the *action* of the light, taken from the luminous origin of the system to the point x, y, z; the word *action* being used in the same sense as in that known law, which is called the law of least action. We may therefore give the name of *surfaces of constant action*, to that series of surfaces for each of which the characteristic function V is equal to some constant quantity, and which have for their differential equation,

$$\delta V = 0 = \frac{\delta v}{\delta \alpha}\,\delta x + \frac{\delta v}{\delta \beta}\,\delta y + \frac{\delta v}{\delta \gamma}\,\delta z. \tag{U$''$}$$

In like manner, if we denote by V' the analogous characteristic function of one of those focal systems considered in the eleventh number, which have their light of the same kind and in the same medium, but converging towards or diverging from one focus; the general expression of this function V' will be $V' = v\rho + \text{const.}$, ρ being the distance from the focus; and the differential equation

$$\delta\,.\,v\rho = 0 = \delta V', \tag{V$''$}$$

will represent a series of surfaces, which are analogous to the surfaces (U$''$). In the case of ordinary light, these surfaces (V$''$) are spheres, and they may called in general, *spheroids of constant action*; the focus of the focal system being called the *centre* of the spheroid. The general equation of such a spheroid contains four arbitrary constants, of which three are the coordinates of the centre; and if we determine these four constants, by the condition that for some given values of x, y, z, that is for some given point of a given system, certain first terms of the development

$$V' = \delta V' + \tfrac{1}{2}\delta^2 V' + \&\text{c.}$$

may be equal to the corresponding terms of the development

$$V = \delta V + \tfrac{1}{2}\delta^2 V + \&\text{c.}$$

the spheroid thus determined will be an *osculating spheroid*, to the surface of constant action which passes through the given point of the system. The conditions

$$V' = V, \quad \delta V' = \delta V, \tag{W$''$}$$

may be satisfied independently of the ratios of δx, δy, δz, by taking the centre of the spheroid any where upon the given ray, that is, by establishing between the three coordinates of this centre the two equations of the ray, and by assigning a proper value to the other arbitrary constant; there still remains therefore, after satisfying the conditions (W$''$), an arbitrary parameter depending on the position of the centre, which we may determine by the equation,

$$\delta^2 V' = \delta^2 V, \tag{X$''$}$$

assigning any arbitrary ratios to the three variations δx, δy, δz, or rather any value to the one ratio

$$\frac{\gamma\,\delta x - \alpha\,\delta z}{\gamma\,\delta y - \beta\,\delta z};$$

because, by the relations (H),

$$\delta^2 V = \frac{\delta^2 V}{\delta x^2}\left(\delta x - \frac{\alpha}{\gamma}\,\delta z\right)^2 + 2\frac{\delta^2 V}{\delta x\,\delta y}\left(\delta x - \frac{\alpha}{\gamma}\,\delta z\right)\left(\delta y - \frac{\beta}{\gamma}\,\delta z\right) + \frac{\delta^2 V}{\delta y^2}\left(\delta y - \frac{\beta}{\gamma}\,\delta z\right)^2,$$

so that the condition (X″) may be thus written:

$$\begin{aligned} 0 = {} & \left(\frac{\delta^2 V'}{\delta x^2} - \frac{\delta^2 V}{\delta x^2}\right)\left(\delta x - \frac{\alpha}{\gamma}\,\delta z\right)^2 \\ & + 2\left(\frac{\delta^2 V'}{\delta x\,\delta y} - \frac{\delta^2 V}{\delta x\,\delta y}\right)\left(\delta x - \frac{\alpha}{\gamma}\,\delta z\right)\left(\delta y - \frac{\beta}{\gamma}\,\delta z\right) \\ & + \left(\frac{\delta^2 V'}{\delta y^2} - \frac{\delta^2 V}{\delta y^2}\right)\left(\delta y - \frac{\beta}{\gamma}\,\delta z\right)^2, \end{aligned} \tag{Y″}$$

or, by a further transformation,

$$\begin{aligned} 0 = {} & \left(\frac{1}{\rho}\cdot\frac{\delta^2 v}{\delta\alpha^2} - \frac{\delta^2 V}{\delta x^2}\right)\left(\delta x - \frac{\alpha}{\gamma}\,\delta z\right)^2 \\ & + 2\left(\frac{1}{\rho}\cdot\frac{\delta^2 v}{\delta\alpha\,\delta\beta} - \frac{\delta^2 V}{\delta x\,\delta y}\right)\left(\delta x - \frac{\alpha}{\gamma}\,\delta z\right)\left(\delta y - \frac{\beta}{\gamma}\,\delta z\right) \\ & + \left(\frac{1}{\rho}\cdot\frac{\delta^2 v}{\delta\beta^2} - \frac{\delta^2 V}{\delta y^2}\right)\left(\delta y - \frac{\beta}{\gamma}\,\delta z\right)^2, \end{aligned} \tag{Z″}$$

ρ being here the distance of the point $x\ y\ z$ upon the ray, beyond the centre of the spheroid. This equation (Z″) contains the law of osculation of the spheroid, since it expresses the dependence of the distance ρ on the direction of the plane passing through the ray and through the consecutive point $x + \delta x$, $y + \delta y$, $z + \delta z$. We shall call this plane the plane of osculation of the spheroid; and we see, by comparing (Z″) with (C′), that the extreme values of ρ correspond to those directions of the plane of osculation in which it touches the developable pencils; while the corresponding extreme positions of the centre of the osculating spheroid, are contained upon the caustic surfaces. And when the ray is one of those principal rays determined in the preceding number, it is easy to prove that the equation (Z″) is satisfied independently of the ratios of the differentials, if we assign to ρ the value which belongs to the principal focus; the principal foci are therefore the centres of spheroids, which have complete contact of the second order with the surfaces of constant action. The equations which express this property of the principal foci are

$$\left.\begin{aligned} \frac{1}{\rho}\frac{\delta^2 v}{\delta\alpha^2} = \frac{\delta^2 V}{\delta x^2}, \quad \frac{1}{\rho}\frac{\delta^2 v}{\delta\alpha\,\delta\beta} = \frac{\delta^2 V}{\delta x\,\delta y}, \quad \frac{1}{\rho}\frac{\delta^2 v}{\delta\beta^2} = \frac{\delta^2 V}{\delta y^2}, \\ \frac{1}{\rho}\frac{\delta^2 v}{\delta\alpha\,\delta\gamma} = \frac{\delta^2 V}{\delta x\,\delta z}, \quad \frac{1}{\rho}\frac{\delta^2 v}{\delta\beta\,\delta\gamma} = \frac{\delta^2 V}{\delta y\,\delta z}, \quad \frac{1}{\rho}\frac{\delta^2 v}{\delta\gamma^2} = \frac{\delta^2 V}{\delta z^2}, \end{aligned}\right\} \tag{A‴}$$

of which any three include the rest; they may also be thus written,

$$\left.\begin{array}{lll}
\dfrac{\delta^2 V'}{\delta x^2} = \dfrac{\delta^2 V}{\delta x^2}, & \dfrac{\delta^2 V'}{\delta y^2} = \dfrac{\delta^2 V}{\delta y^2}, & \dfrac{\delta^2 V'}{\delta z^2} = \dfrac{\delta^2 V}{\delta z^2}, \\
\dfrac{\delta^2 V'}{\delta x\, \delta y} = \dfrac{\delta^2 V}{\delta x\, \delta y}, & \dfrac{\delta^2 V'}{\delta y\, \delta z} = \dfrac{\delta^2 V}{\delta y\, \delta z}, & \dfrac{\delta^2 V'}{\delta z\, \delta x} = \dfrac{\delta^2 V}{\delta z\, \delta x};
\end{array}\right\} \qquad \text{(B}'''\text{)}$$

and may be summed up in the one equation (X″), by considering δx, δy, δz, as independent. With respect to those rays which are not the principal rays of the system, and for which the equation (X″) can only be satisfied by assigning some particular value to the ratio

$$\frac{\gamma\, \delta x - \alpha\, \delta z}{\gamma\, \delta y - \beta\, \delta z},$$

that is, some particular position to the plane of osculation of the spheroid, we find, by reasonings similar to those of the twelth number, the following law of osculation:

$$\frac{\dfrac{1}{\rho_1} - \dfrac{1}{\rho}}{\dfrac{1}{\rho} - \dfrac{1}{\rho_2}} = \zeta \left(\frac{\sin.(\psi - \psi_1)}{\sin.(\psi_2 - \psi)} \right)^2 : \qquad \text{(C}'''\text{)}$$

ρ_1, ρ_2, being the extreme values of ρ; ψ_1, ψ_2, the corresponding values of the angle ψ, comprised between the plane of osculation and any fixed plane that passes through the ray; and the coefficient ζ being independent of ρ and ψ, and having the same meaning as before. The formula (C‴) may be written in the following manner:

$$\frac{\rho - \rho_1}{\rho_2 - \rho} = \frac{\zeta \rho_1}{\rho_2} \cdot \left(\frac{\sin.(\psi - \psi_1)}{\sin.(\psi_2 - \psi)} \right)^2 : \qquad \text{(D}'''\text{)}$$

in this kind of osculation, therefore, as in the former, the distances of the variable focus or centre from the points where the ray touches the two caustic surfaces, are proportional to the squares of the sines of the angles which the plane of osculation makes with the tangent planes to the developable pencils.

On Osculating Focal Reflectors or Refractors.

16. Besides the two preceding kinds of osculation, it is interesting to consider a third kind, which exists between the last reflecting or refracting surface, and certain other surfaces, which would have reflected or refracted to or from one focus the rays of the last incident system, and which we shall therefore call *focal reflectors or refractors.* Let V_1, V_2, denote, as in the sixth number, any two successive forms of the characteristic function V, of which we shall suppose that V_2 belongs to the system in its given state, and V_1 to the same system before its last reflexion or refraction; then, by the number cited, the equation $V_1 - V_2 = 0$, will be a form for the equation of the reflector or refractor, at which the state of the system was last changed, and which we shall consider as known. Let V_2' be the form which V_2 would have,

if the rays of the final system all converged or diverged from one focus, this form being such as was assigned in the fifteenth number, and depending only on the nature of the light and of the final medium, but involving four arbitrary constants, of which three are the coordinates of the focus; then it is easy to prove that the equation with four arbitrary constants, of the focal surface, which would have reflected or refracted to or from one focus the rays of the last incident system, is

$$V_1 - V_2' = 0. \tag{E$'''$}$$

We may determine the four arbitrary constants of V_2' in this equation, by the condition that the focal reflector or refractor shall touch the given reflector or refractor at a given point, and osculate in a given direction. The condition of contact, of the first order, is expressed by the equations

$$V_2 = V_2', \quad \delta V_2 = \delta V_2', \tag{F$'''$}$$

and may be satisfied by establishing between the three coordinates of the focus the two equations of the ray, and by assigning a proper value to the remaining arbitrary constant; and the position of the focus upon the given ray, may be determined by the condition of osculation in the given direction, which is expressed by the equation

$$\delta^2 V_2 = \delta^2 V_2', \tag{G$'''$}$$

assigning the given ratios to the variations δx, δy, δz. This equation (G$'''$) being the same with that marked (X$''$) in the foregoing number, we can deduce from it the same consequences; the osculation therefore between the focal surface (E$'''$) and the given reflector or refractor, follows the same law as the osculation between the spheroid of constant action (V$''$) and the given surface (U$''$) for which the function V is constant; in such a manner that the focus of the focal reflector or refractor coincides with the centre of the spheroid, if the point of contact, and the plane of osculation be the same. The distances therefore of the focus of the focal reflector or refractor from the points in which the ray touches the two caustic surfaces, are proportional to the squares of the sines of the angles which the plane of osculation makes with the tangent planes to the two developable pencils. And when the ray is one of those principal rays, assigned in the fourteenth number, (the focus of the focal surface being at the principal focus corresponding,) then the contact of the second order is most complete, and the two reflectors or refractors osculate to each other in all directions.

On Foci by Projection, and Virtual Foci.

17. Another kind of focus, of which the law is similar, though not the same, may be deduced in the following manner. If we conceive a plane passing through a given ray of a given optical system, and through a point infinitely near to this given ray; the ray which passes through the near point may be projected on the plane, and the intersection of its projection with the given ray may be called a *focus by projection*. Suppose, to simplify the first calculations, that the given ray is the axis of z, and that the infinitely near point is contained in the plane of xy; its coordinates in this plane being denoted by δx, δy, and the cosines of the angles which the near ray makes with the axes of x and y, being $\delta\alpha$, $\delta\beta$: then,

if we denote the general coordinates of this near ray by $x_{\prime\prime}$ $y_{\prime\prime}$ $z_{\prime\prime}$, its equations may be thus written,

$$x_{\prime\prime} = \delta x + z_{\prime\prime}\,\delta\alpha, \quad y_{\prime\prime} = \delta y + z_{\prime\prime}\,\delta\beta, \tag{H$'''$}$$

and the connexions between δx, δy, $\delta\alpha$, $\delta\beta$, will be expressed by the two following conditions:

$$\left.\begin{aligned}
\frac{\delta^2 V}{\delta x^2}\,\delta x + \frac{\delta^2 V}{\delta x\,\delta y}\,\delta y &= \frac{\delta^2 v}{\delta\alpha^2}\,\delta\alpha + \frac{\delta^2 v}{\delta\alpha\,\delta\beta}\,\delta\beta, \\
\frac{\delta^2 V}{\delta x\,\delta y}\,\delta x + \frac{\delta^2 V}{\delta y^2}\,\delta y &= \frac{\delta^2 v}{\delta\alpha\,\delta\beta}\,\delta\alpha + \frac{\delta^2 v}{\delta\beta^2}\,\delta\beta,
\end{aligned}\right\} \tag{I$'''$}$$

which are obtained by differentiating (C) and making $\delta z = 0$, $\delta\gamma = 0$. The equation of the plane on which the near ray (H$'''$) is to be projected, may be put under the form

$$\frac{y_{\prime\prime}}{x_{\prime\prime}} = \frac{\delta y}{\delta x}; \tag{K$'''$}$$

and if p be the vertical ordinate of the focus by projection, the equation of the projecting plane is

$$\frac{y_{\prime\prime} - \delta y - z_{\prime\prime}\,\delta\beta}{x_{\prime\prime} - \delta x - z_{\prime\prime}\,\delta\alpha} = \frac{\delta y + p\,.\,\delta\beta}{\delta x + p\,.\,\delta\alpha}, \tag{L$'''$}$$

p being determined by the condition that the two planes (K$'''$) (L$'''$) shall be perpendicular to each other, which gives

$$\frac{1}{p} = -\frac{\delta x\,\delta\alpha + \delta y\,\delta\beta}{\delta x^2 + \delta y^2}. \tag{M$'''$}$$

In general, whatever arbitrary position we assign to the rectangular axes, if we represent by $x + \alpha p$, $y + \beta p$, $z + \gamma p$, the coordinates of the focus by projection, those of the given point being x, y, z, and those of the near point $x + \delta x$, $y + \delta y$, $z + \delta z$, we shall find by a similar process,

$$-\frac{1}{p} = \frac{\delta\alpha\,\delta x' + \delta\beta\,\delta y' + \delta\gamma\,\delta z'}{\delta x'^2 + \delta y'^2 + \delta z'^2} = \frac{\delta\alpha\,\delta x + \delta\beta\,\delta y + \delta\gamma\,\delta z}{\delta x^2 + \delta y^2 + \delta z^2 - (\alpha\,\delta x + \beta\,\delta y + \gamma\,\delta z)^2}, \tag{N$'''$}$$

$\delta x'$, $\delta y'$, $\delta z'$, having the same meanings as in the fifth number. And since the equations (C) give, by differentiation and elimination,

$$\left.\begin{aligned}
v''\,\delta\alpha &= \left(\frac{\delta^2 v}{\delta\alpha^2} + \frac{\delta^2 v}{\delta\beta^2} + \frac{\delta^2 v}{\delta\gamma^2}\right)\delta\frac{\delta V}{\delta x} - \left(\frac{\delta^2 v}{\delta\alpha^2}\,\delta\frac{\delta V}{\delta x} + \frac{\delta^2 v}{\delta\alpha\,\delta\beta}\,\delta\frac{\delta V}{\delta y} + \frac{\delta^2 v}{\delta\alpha\,\delta\gamma}\,\delta\frac{\delta V}{\delta z}\right), \\
v''\,\delta\beta &= \left(\frac{\delta^2 v}{\delta\alpha^2} + \frac{\delta^2 v}{\delta\beta^2} + \frac{\delta^2 v}{\delta\gamma^2}\right)\delta\frac{\delta V}{\delta y} - \left(\frac{\delta^2 v}{\delta\alpha\,\delta\beta}\,\delta\frac{\delta V}{\delta x} + \frac{\delta^2 v}{\delta\beta^2}\,\delta\frac{\delta V}{\delta y} + \frac{\delta^2 v}{\delta\beta\,\delta\gamma}\,\delta\frac{\delta V}{\delta z}\right), \\
v''\,\delta\gamma &= \left(\frac{\delta^2 v}{\delta\alpha^2} + \frac{\delta^2 v}{\delta\beta^2} + \frac{\delta^2 v}{\delta\gamma^2}\right)\delta\frac{\delta V}{\delta z} - \left(\frac{\delta^2 v}{\delta\alpha\,\delta\gamma}\,\delta\frac{\delta V}{\delta x} + \frac{\delta^2 v}{\delta\beta\,\delta\gamma}\,\delta\frac{\delta V}{\delta y} + \frac{\delta^2 v}{\delta\gamma^2}\,\delta\frac{\delta V}{\delta z}\right),
\end{aligned}\right\} \tag{O$'''$}$$

and therefore

$$v''(\delta\alpha\,\delta x + \delta\beta\,\delta y + \delta\gamma\,\delta z) = \left(\frac{\delta^2 v}{\delta\alpha^2} + \frac{\delta^2 v}{\delta\beta^2} + \frac{\delta^2 v}{\delta\gamma^2}\right)\delta^2 V$$
$$- \left(\delta'\frac{\delta v}{\delta\alpha}\,\delta\frac{\delta V}{\delta x} + \delta'\frac{\delta v}{\delta\beta}\,\delta\frac{\delta V}{\delta y} + \delta'\frac{\delta v}{\delta\gamma}\,\delta\frac{\delta V}{\delta z}\right), \qquad (\mathrm{P}''')$$

we find, finally,

$$\frac{v''}{p}(\delta x'^2 + \delta y'^2 + \delta z'^2) = \delta\frac{\delta V}{\delta x}\,\delta'\frac{\delta v}{\delta\alpha} + \delta\frac{\delta V}{\delta y}\,\delta'\frac{\delta v}{\delta\beta} + \delta\frac{\delta V}{\delta z}\,\delta'\frac{\delta v}{\delta\gamma}$$
$$- \left(\frac{\delta^2 v}{\delta\alpha^2} + \frac{\delta^2 v}{\delta\beta^2} + \frac{\delta^2 v}{\delta\gamma^2}\right)\delta^2 V : \qquad (\mathrm{Q}''')$$

v'' being the same function as before. It results from this equation (Q''') or from (M''') and (I''') that when the given ray is taken for the axis of z we shall have

$$\frac{v''}{p} = \left(\frac{\delta^2 v}{\delta\alpha\,\delta\beta}\frac{\delta^2 V}{\delta x\,\delta y} - \frac{\delta^2 v}{\delta\beta^2}\frac{\delta^2 V}{\delta x^2}\right)(\cos.\Pi)^2 + \left(\frac{\delta^2 v}{\delta\alpha\,\delta\beta}\frac{\delta^2 V}{\delta x\,\delta y} - \frac{\delta^2 v}{\delta\alpha^2}\frac{\delta^2 V}{\delta y^2}\right)(\sin.\Pi)^2$$
$$+ \left\{\frac{\delta^2 v}{\delta\alpha\,\delta\beta}\left(\frac{\delta^2 V}{\delta x^2} + \frac{\delta^2 V}{\delta y^2}\right) - \frac{\delta^2 V}{\delta x\,\delta y}\left(\frac{\delta^2 v}{\delta\alpha^2} + \frac{\delta^2 v}{\delta\beta^2}\right)\right\}\sin.\Pi\,\cos.\Pi, \qquad (\mathrm{R}''')$$

if we put $\delta y = \delta x \tan.\Pi$, so that Π denotes the angle which the plane of projection makes with the plane of xz. Differentiating (R''') for Π only, we find that the values of this angle which correspond to the extreme positions of the focus by projection are determined by the condition

$$\left(\frac{\delta^2 v}{\delta\alpha^2}\frac{\delta^2 V}{\delta y^2} - \frac{\delta^2 v}{\delta\beta^2}\frac{\delta^2 V}{\delta x^2}\right)\tan.2\Pi = \frac{\delta^2 v}{\delta\alpha\,\delta\beta}\left(\frac{\delta^2 V}{\delta x^2} + \frac{\delta^2 V}{\delta y^2}\right) - \frac{\delta^2 V}{\delta x\,\delta y}\left(\frac{\delta^2 v}{\delta\alpha^2} + \frac{\delta^2 v}{\delta\beta^2}\right) : \qquad (\mathrm{S}''')$$

the *planes of extreme projection*, that is, the planes which correspond to the extreme values of p, are therefore perpendicular to each other; and if we suppose them taken for the planes of xz, yz, and denote by p_1, p_2, the corresponding values of p, we shall have

$$\left.\begin{aligned} 0 &= \frac{\delta^2 v}{\delta\alpha\,\delta\beta}\left(\frac{\delta^2 V}{\delta x^2} + \frac{\delta^2 V}{\delta y^2}\right) - \frac{\delta^2 V}{\delta x\,\delta y}\left(\frac{\delta^2 v}{\delta\alpha^2} + \frac{\delta^2 v}{\delta\beta^2}\right), \\ \frac{v''}{p_1} &= \frac{\delta^2 v}{\delta\alpha\,\delta\beta}\frac{\delta^2 V}{\delta x\,\delta y} - \frac{\delta^2 v}{\delta\beta^2}\frac{\delta^2 V}{\delta x^2}, \quad \frac{v''}{p_2} = \frac{\delta^2 v}{\delta\alpha\,\delta\beta}\frac{\delta^2 V}{\delta x\,\delta y} - \frac{\delta^2 v}{\delta\alpha^2}\frac{\delta^2 V}{\delta y^2}, \end{aligned}\right\} \qquad (\mathrm{T}''')$$

and finally the dependence of p upon Π, that is, the law of the focus by projection will be expressed by the following formula:

$$\frac{1}{p} = \frac{1}{p_1}(\cos.\Pi)^2 + \frac{1}{p_2}(\sin.\Pi)^2. \qquad (\mathrm{U}''')$$

When the given ray is one of those principal rays determined in the foregoing numbers, the angle Π disappears from this formula, and all the foci by projection coincide in the principal focus, the condition (S$'''$) being at the same time identically satisfied, and failing to determine the planes of extreme projection: but in general these planes can be determined by that condition, and have a remarkable connexion with the tangent planes to the developable pencils, which can be deduced from the equation (L$'$) of the ninth number,

$$\delta' \frac{\delta v}{\delta \alpha} \, \delta \frac{\delta V}{\delta y} = \delta' \frac{\delta v}{\delta \beta} \, \delta \frac{\delta V}{\delta x}.$$

For, when we suppose $\delta z = 0$, $\delta y = \delta x \tan. \Pi$, we find from this equation (L$'$) the following quadratic equation to determine the two values of $\tan. \Pi$ corresponding to the tangent planes of the two developable pencils:

$$\begin{aligned} 0 = \frac{\delta^2 v}{\delta \alpha^2} \frac{\delta^2 V}{\delta x \, \delta y} - \frac{\delta^2 v}{\delta \alpha \, \delta \beta} \frac{\delta^2 V}{\delta x^2} &+ \left(\frac{\delta^2 v}{\delta \alpha \, \delta \beta} \frac{\delta^2 V}{\delta y^2} - \frac{\delta^2 v}{\delta \beta^2} \frac{\delta^2 V}{\delta x \, \delta y} \right) (\tan. \Pi)^2 \\ &+ \left(\frac{\delta^2 v}{\delta \alpha^2} \frac{\delta^2 V}{\delta y^2} - \frac{\delta^2 v}{\delta \beta^2} \frac{\delta^2 V}{\delta x^2} \right) \tan. \Pi : \end{aligned} \tag{V$'''$}$$

and if the first condition (T$'''$) be satisfied, that is, if the planes of extreme projection be taken for the planes of xz, yz, the product of the two values of $\tan. \Pi$ determined by this quadratic will be unity; the tangent planes to the developable pencils are therefore symmetrically situated with respect to the planes of extreme projection, the bisectors of the angles formed by the one pair of planes bisecting also the angles of the other pair. The tangent planes to the developable pencils are not always perpendicular to each other, and therefore are not always fit to be taken for rectangular coordinate planes, however remarkable they may be in other respects; but the planes of extreme projection, determined in the present number, possess this important property, and may be considered as furnishing for any given straight ray of an optical system, ordinary or extraordinary, (except the principal rays,) two *natural coordinate planes*, which contain the given ray, and are perpendicular to each other. And whenever the developable pencils are also perpendicular to each other, the tangent planes to these pencils will coincide with the planes of extreme projection, and the extreme foci by projection will be contained upon the caustic surfaces. This perpendicularity of the developable pencils requires that there should exist a series of surfaces perpendicular to the rays of the system, and having for their differential equation

$$\alpha \, \delta x + \beta \, \delta y + \gamma \, \delta z = 0; \tag{W$'''$}$$

and reciprocally when this equation is integrable, the perpendicularity here spoken of, exists, and we shall say that the system is *rectangular*. This condition is satisfied in the case of ordinary systems, because, for such systems, the differential equation (U$''$) of the surfaces of constant action becomes

$$\delta V = m(\alpha \, \delta x + \beta \, \delta y + \gamma \, \delta z) = 0,$$

and consequently coincides with the equation (W$'''$), m having the same meaning as in the thirteenth number; the rays of an ordinary system are therefore perpendicular to the surfaces for which the function V is constant, and their planes of extreme projection are touched by the developable pencils. We may also remark that for such systems $\zeta = 1$, and the osculating foci coincide with the foci by projection.

18. There is yet another kind of foci which we shall call *Virtual Foci*, and which it may be interesting to consider, because they conduct to the same pair of natural coordinate planes as those which we have deduced in the foregoing number, and because they furnish new applications of the characteristic functions of the system. By a *virtual focus* of a given ray, we shall understand a point in which it is nearest to an infinitely near ray of the system. To explain this more fully, let us observe, that if we establish any arbitrary relation between α, β, γ, distinct from the relation $\alpha^2 + \beta^2 + \gamma^2 = 1$, we shall obtain some corresponding relation between

$$\frac{\delta V}{\delta x}, \quad \frac{\delta V}{\delta y}, \quad \frac{\delta V}{\delta z},$$

by eliminating α, β, γ, between the equations (C); the result of this elimination, which we may represent by

$$F\left(\frac{\delta V}{\delta x}, \frac{\delta V}{\delta y}, \frac{\delta V}{\delta z}\right) = 0,$$

F denoting an arbitrary function, will be the equation of a *pencil*, that is of a surface of right lines, composed by rays of the system: and unless this surface be one of the developable pencils determined in the ninth number, the rays of which it is composed will not intersect consecutively, so that there will be only a *virtual intersection*, or nearest approach, even between two infinitely near rays. To find the coordinates of this virtual intersection, we are to seek the minimum of $\delta x^2 + \delta y^2 + \delta z^2$, or of $\delta x'^2 + \delta y'^2 + \delta z'^2$, corresponding to given values of α, β, γ, $\delta\alpha$, $\delta\beta$, $\delta\gamma$. Now if we put $r = \alpha x + \beta y + \gamma z$, we shall have

$$\left.\begin{array}{lll} x = x_{\prime} + \alpha r, & y = y_{\prime} + \beta r, & z = z_{\prime} + \gamma r, \\ \delta x = \delta x_{\prime} + \delta\,.\,\alpha r, & \delta y = \delta y_{\prime} + \delta\,.\,\beta r, & \delta z = \delta z_{\prime} + \delta\,.\,\gamma r, \end{array}\right\} \qquad (\mathrm{X}''')$$

and therefore

$$\left.\begin{array}{l} \delta x' = r\,\delta\alpha + \delta x_{\prime} - \alpha(\alpha\,\delta x_{\prime} + \beta\,\delta y_{\prime} + \gamma\,\delta z_{\prime}), \\ \delta y' = r\,\delta\beta + \delta y_{\prime} - \beta(\alpha\,\delta x_{\prime} + \beta\,\delta y_{\prime} + \gamma\,\delta z_{\prime}), \\ \delta z' = r\,\delta\gamma + \delta z_{\prime} - \gamma(\alpha\,\delta x_{\prime} + \beta\,\delta y_{\prime} + \gamma\,\delta z_{\prime}), \end{array}\right\} \qquad (\mathrm{Y}''')$$

$x_{\prime}$, $y_{\prime}$, $z_{\prime}$, and $\delta x'$, $\delta y'$, $\delta z'$, having the same meanings as before; and the condition of minimum gives

$$r = -\frac{\delta\alpha\,\delta x_{\prime} + \delta\beta\,\delta y_{\prime} + \delta\gamma\,\delta z_{\prime}}{\delta\alpha^2 + \delta\beta^2 + \delta\gamma^2}, \qquad (\mathrm{Z}''')$$

which may also be thus written

$$0 = \delta\alpha\,\delta x' + \delta\beta\,\delta y' + \delta\gamma\,\delta z' = \delta\alpha\,\delta x + \delta\beta\,\delta y + \delta\gamma\,\delta z : \qquad (\mathrm{A}^4)$$

or, by the foregoing number,

$$\left(\frac{\delta^2 v}{\delta\alpha^2} + \frac{\delta^2 v}{\delta\beta^2} + \frac{\delta^2 v}{\delta\gamma^2}\right)\delta^2 V = \delta'\frac{\delta v}{\delta\alpha}\,\delta\frac{\delta V}{\delta x} + \delta'\frac{\delta v}{\delta\beta}\,\delta\frac{\delta V}{\delta y} + \delta'\frac{\delta v}{\delta\gamma}\,\delta\frac{\delta V}{\delta z}. \qquad (\mathrm{B}^4)$$

Another transformation of this condition, which shall involve the function W instead of V, may be obtained in the following manner. Let $W_{\prime}$ be the form which the characteristic function W would have, for a system of rays of the same light and in the same medium, but all converging towards or diverging from the one point $x_{\prime}$, $y_{\prime}$, $z_{\prime}$; so that, by the theory already given,

$$\left.\begin{aligned}
\delta\frac{\delta W_{\prime}}{\delta\alpha} &= x_{\prime}\,\delta\frac{\delta^2 v}{\delta\alpha^2} + y_{\prime}\,\delta\frac{\delta^2 v}{\delta\alpha\,\delta\beta} + z_{\prime}\,\delta\frac{\delta^2 v}{\delta\alpha\,\delta\gamma},\\
\delta\frac{\delta W_{\prime}}{\delta\beta} &= x_{\prime}\,\delta\frac{\delta^2 v}{\delta\alpha\,\delta\beta} + y_{\prime}\,\delta\frac{\delta^2 v}{\delta\beta^2} + z_{\prime}\,\delta\frac{\delta^2 v}{\delta\beta\,\delta\gamma},\\
\delta\frac{\delta W_{\prime}}{\delta\gamma} &= x_{\prime}\,\delta\frac{\delta^2 v}{\delta\alpha\,\delta\gamma} + y_{\prime}\,\delta\frac{\delta^2 v}{\delta\beta\,\delta\gamma} + z_{\prime}\,\delta\frac{\delta^2 v}{\delta\gamma^2}:
\end{aligned}\right\} \tag{C^4}$$

then, by differentiating the equations (G′), and attending to the formulæ (Y‴), we find

$$\left.\begin{aligned}
\delta\frac{\delta(W - W_{\prime})}{\delta\alpha} &= \frac{\delta^2 v}{\delta\alpha^2}(\delta x' - r\,\delta\alpha) + \frac{\delta^2 v}{\delta\alpha\,\delta\beta}(\delta y' - r\,\delta\beta) + \frac{\delta^2 v}{\delta\alpha\,\delta\gamma}(\delta z' - r\,\delta\gamma),\\
\delta\frac{\delta(W - W_{\prime})}{\delta\beta} &= \frac{\delta^2 v}{\delta\alpha\,\delta\beta}(\delta x' - r\,\delta\alpha) + \frac{\delta^2 v}{\delta\beta^2}(\delta y' - r\,\delta\beta) + \frac{\delta^2 v}{\delta\beta\,\delta\gamma}(\delta z' - r\,\delta\gamma),\\
\delta\frac{\delta(W - W_{\prime})}{\delta\gamma} &= \frac{\delta^2 v}{\delta\alpha\,\delta\gamma}(\delta x' - r\,\delta\alpha) + \frac{\delta^2 v}{\delta\beta\,\delta\gamma}(\delta y' - r\,\delta\beta) + \frac{\delta^2 v}{\delta\gamma^2}(\delta z' - r\,\delta\gamma),\\
0 &= \alpha(\delta x' - r\,\delta\alpha) + \beta(\delta y' - r\,\delta\beta) + \gamma(\delta z' - r\,\delta\gamma);
\end{aligned}\right\} \tag{D^4}$$

and therefore

$$\left.\begin{aligned}
v''(\delta x' - r\,\delta\alpha) &= \left(\frac{\delta^2 v}{\delta\alpha^2} + \frac{\delta^2 v}{\delta\beta^2} + \frac{\delta^2 v}{\delta\gamma^2}\right)\,\delta\frac{\delta(W - W_{\prime})}{\delta\alpha}\\
&\quad - \left\{\frac{\delta^2 v}{\delta\alpha^2}\,\delta\frac{\delta(W - W_{\prime})}{\delta\alpha} + \frac{\delta^2 v}{\delta\alpha\,\delta\beta}\,\delta\frac{\delta(W - W_{\prime})}{\delta\beta} + \frac{\delta^2 v}{\delta\alpha\,\delta\gamma}\,\delta\frac{\delta(W - W_{\prime})}{\delta\gamma}\right\},\\
v''(\delta x' - r\,\delta\beta) &= \left(\frac{\delta^2 v}{\delta\alpha^2} + \frac{\delta^2 v}{\delta\beta^2} + \frac{\delta^2 v}{\delta\gamma^2}\right)\,\delta\frac{\delta(W - W_{\prime})}{\delta\beta}\\
&\quad - \left\{\frac{\delta^2 v}{\delta\alpha\,\delta\beta}\,\delta\frac{\delta(W - W_{\prime})}{\delta\alpha} + \frac{\delta^2 v}{\delta\beta^2}\,\delta\frac{\delta(W - W_{\prime})}{\delta\beta} + \frac{\delta^2 v}{\delta\beta\,\delta\gamma}\,\delta\frac{\delta(W - W_{\prime})}{\delta\gamma}\right\},\\
v''(\delta x' - r\,\delta\gamma) &= \left(\frac{\delta^2 v}{\delta\alpha^2} + \frac{\delta^2 v}{\delta\beta^2} + \frac{\delta^2 v}{\delta\gamma^2}\right)\,\delta\frac{\delta(W - W_{\prime})}{\delta\gamma}\\
&\quad - \left\{\frac{\delta^2 v}{\delta\alpha\,\delta\gamma}\,\delta\frac{\delta(W - W_{\prime})}{\delta\alpha} + \frac{\delta^2 v}{\delta\beta\,\delta\gamma}\,\delta\frac{\delta(W - W_{\prime})}{\delta\beta} + \frac{\delta^2 v}{\delta\gamma^2}\,\delta\frac{\delta(W - W_{\prime})}{\delta\gamma}\right\}.
\end{aligned}\right\} \tag{E^4}$$

By these equations the conditions (A^4) may be transformed into the following:

$$\left.\begin{aligned}
&v''r(\delta\alpha^2 + \delta\beta^2 + \delta\gamma^2) + \left(\frac{\delta^2 v}{\delta\alpha^2} + \frac{\delta^2 v}{\delta\beta^2} + \frac{\delta^2 v}{\delta\gamma^2}\right)\,\delta^2(W - W_{\prime})\\
&\quad = \delta\frac{\delta v}{\delta\alpha}\,\delta\frac{\delta(W - W_{\prime})}{\delta\alpha} + \delta\frac{\delta v}{\delta\beta}\,\delta\frac{\delta(W - W_{\prime})}{\delta\beta} + \delta\frac{\delta v}{\delta\gamma}\,\delta\frac{\delta(W - W_{\prime})}{\delta\gamma}.
\end{aligned}\right\} \tag{F^4}$$

To find the geometrical law expressed by this last formula, let us take the given ray for the axis of z, and let us choose the planes of xz, yz, in such a manner that the bisectors of their angles shall bisect also the angles formed by the developable pencils; we shall then have, by the fourteenth number, $E = E''$, that is,

$$\frac{\delta^2 v}{\delta\alpha\,\delta\beta}\left(\frac{\delta^2 W}{\delta\alpha^2} + \frac{\delta^2 W}{\delta\beta^2}\right) = \frac{\delta^2 W}{\delta\alpha\,\delta\beta}\left(\frac{\delta^2 v}{\delta\alpha^2} + \frac{\delta^2 v}{\delta\beta^2}\right), \tag{G^4}$$

and the formula (F^4) will become

$$v''r = \left(\frac{\delta^2 v}{\delta\alpha\,\delta\beta}\frac{\delta^2 W}{\delta\alpha\,\delta\beta} - \frac{\delta^2 v}{\delta\beta^2}\frac{\delta^2 W}{\delta\alpha^2}\right)\frac{\delta\alpha^2}{\delta\alpha^2 + \delta\beta^2} + \left(\frac{\delta^2 v}{\delta\alpha\,\delta\beta}\frac{\delta^2 W}{\delta\alpha\,\delta\beta} - \frac{\delta^2 v}{\delta\alpha^2}\frac{\delta^2 W}{\delta\beta^2}\right)\frac{\delta\beta^2}{\delta\alpha^2 + \delta\beta^2}; \tag{H^4}$$

or finally

$$r = r_1(\cos.\omega)^2 + r_2(\sin.\omega)^2, \tag{I^4}$$

when we put

$$v''r_1 = \frac{\delta^2 v}{\delta\alpha\,\delta\beta}\frac{\delta^2 W}{\delta\alpha\,\delta\beta} - \frac{\delta^2 v}{\delta\beta^2}\frac{\delta^2 W}{\delta\alpha^2}, \quad v''r_2 = \frac{\delta^2 v}{\delta\alpha\,\delta\beta}\frac{\delta^2 W}{\delta\alpha\,\delta\beta} - \frac{\delta^2 v}{\delta\alpha^2}\frac{\delta^2 W}{\delta\beta^2}, \quad \delta\beta = \delta\alpha\ \tan.\omega: \tag{K^4}$$

ω being the angle which the plane passing through the given ray and parallel to the near ray makes with the plane of xz; and r_1, r_2 being the extreme values of r.

The equation (I^4) expresses in a simple manner the law of the virtual focus. It shews that the extreme positions of that focus correspond to the same pair of *natural coordinate planes*, passing through the given ray, which we considered in the preceding number, and which we may therefore call the *planes of extreme virtual foci*, as well as the *planes of extreme projection.* Indeed, when the given ray is one of the principal rays of the system, assigned in the fourteenth number, then all the virtual foci, as well as all the other foci hitherto considered, coincide in the principal focus: and the planes of extreme virtual foci become, in this case, indeterminate. However, we shall shew that their place is then supplied by another remarkable pair of planes, which pass through the principal ray, and complete the system of natural coordinates: but for this purpose it is necessary to enter briefly on the theory of aberration from a principal focus, which we shall do in the following number.

Aberrations from a Principal Focus.

19. If we conceive a plane cutting a given ray perpendicularly at a given point, this plane will be nearly perpendicular to the near rays, and will cut those rays in points near to the given point: the distances of these near points from the given point, are the *lateral aberrations* of the near rays, and the cutting plane may be called the *plane of aberration.* Let x, y, z, be the coordinates of the given point, and $x + \Delta x$, $y + \Delta y$, $z + \Delta z$, the coordinates of the point in which a near ray is cut by the plane of aberration, Δ being here the mark of a finite difference; we shall have the condition

$$0 = \alpha\,\Delta x + \beta\,\Delta y + \gamma\,\Delta z, \tag{L^4}$$

α, β, γ, being the cosines of the angles which the given ray makes with the axes of x, y, z: and if we determine the successive differentials of x, y, z, with reference to α, β, γ, by differentiating the equations (C) or (K) as if α, β, γ, were three independent variables, and by putting

$$\left.\begin{aligned} 0 &= \alpha\,\delta x + \beta\,\delta y + \gamma\,\delta z, \\ 0 &= \alpha\,\delta^2 x + \beta\,\delta^2 y + \gamma\,\delta^2 z, \\ 0 &= \alpha\,\delta^3 x + \beta\,\delta^3 y + \gamma\,\delta^3 z, \\ &\text{\&c.} \end{aligned}\right\} \tag{M^4}$$

we shall have

$$\left.\begin{aligned} \Delta x &= [\delta x] + \frac{1}{2}[\delta^2 x] + \frac{1}{2\,.\,3}[\delta^3 x] + \text{\&c.}, \\ \Delta y &= [\delta y] + \frac{1}{2}[\delta^2 y] + \frac{1}{2\,.\,3}[\delta^3 y] + \text{\&c.}, \\ \Delta z &= [\delta z] + \frac{1}{2}[\delta^2 z] + \frac{1}{2\,.\,3}[\delta^3 z] + \text{\&c.}, \end{aligned}\right\} \tag{N^4}$$

the expressions $[\delta x]$, $[\delta^2 x]$, &c., being formed from δx, $\delta^2 x$, &c., by changing the differentials $\delta\alpha$, $\delta\beta$, $\delta\gamma$, to the finite differences $\Delta\alpha$, $\Delta\beta$, $\Delta\gamma$: and finally, the lateral aberration of the near ray will have for expression

$$\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}.$$

Let us apply this general theory to the case when the ray from which the aberrations are measured, is a principal ray of the system: and in order to simplify the calculations, let us take this ray for the axis of z, and the principal focus for origin. Then, if we neglect the squares and products of $\Delta\alpha$, $\Delta\beta$, we find by the preceding theory,

$$\Delta x = \rho\,\Delta\alpha, \quad \Delta y = \rho\,\Delta\beta, \quad \Delta z = 0, \tag{O^4}$$

ρ being the distance from the principal focus to the plane of aberration; if, therefore, we suppose this distance ρ to be unity, and represent by a, b, the corresponding values of $\Delta\alpha$, $\Delta\beta$, we shall have,

$$\Delta\alpha = a, \quad \Delta\beta = b; \tag{P^4}$$

and if we take the principal focus for origin, the coordinates of the point in which the near ray intersects the plane of aberration will be a, b, 1. If now we conceive another plane of aberration, perpendicular to the principal ray and passing through the principal focus, we shall have, for this new plane, $\rho = 0$, and the expressions (O^4) for the components of aberration vanish: in this case, therefore, it is necessary to carry the approximation farther, and take account of terms of the second dimension, in the variations of α, β, γ. For this purpose we may differentiate twice successively the equations (K), as if α, β, γ, were independent, making after the differentiations, x, y, z, δx, δy, δz, $\delta^2 z$, each $= 0$, and changing $\delta\alpha$, $\delta\beta$, $\delta\gamma$, $\delta^2 x$, $\delta^2 y$, to $\Delta\alpha$, $\Delta\beta$, $\Delta\gamma$, $2\Delta x$, $2\Delta y$. In this manner we find

$$\left.\begin{aligned} \frac{1}{2}\left[\delta^2\frac{\delta W}{\delta\alpha}\right] &= \frac{\delta^2 v}{\delta\alpha^2}\,\Delta x + \frac{\delta^2 v}{\delta\alpha\,\delta\beta}\,\Delta y, \\ \frac{1}{2}\left[\delta^2\frac{\delta W}{\delta\beta}\right] &= \frac{\delta^2 v}{\delta\alpha\,\delta\beta}\,\Delta x + \frac{\delta^2 v}{\delta\beta^2}\,\Delta y, \end{aligned}\right\} \tag{Q^4}$$

in which we may put

$$\left.\begin{aligned}\left[\delta^2\frac{\delta W}{\delta\alpha}\right] &= \frac{\delta^3 W}{\delta\alpha^3}a^2 + \frac{\delta^3 W}{\delta\alpha^2\,\delta\beta}ab + \frac{\delta^3 W}{\delta\alpha\,\delta\beta^2}b^2,\\ \left[\delta^2\frac{\delta W}{\delta\beta}\right] &= \frac{\delta^3 W}{\delta\alpha^2\,\delta\beta}a^2 + \frac{\delta^3 W}{\delta\alpha\,\delta\beta^2}ab + \frac{\delta^3 W}{\delta\beta^3}b^2,\end{aligned}\right\} \tag{R^4}$$

changing $\Delta\alpha$, $\Delta\beta$, to their expressions (P^4), and observing that the general relation

$$(\alpha+\Delta\alpha)^2 + (\beta+\Delta\beta)^2 + (\gamma+\Delta\gamma)^2 = \alpha^2+\beta^2+\gamma^2 = 1,$$

gives here

$$0 = 2\,\Delta\gamma + (\Delta\alpha)^2 + (\Delta\beta)^2 + (\Delta\gamma)^2,$$

so that the terms $\Delta\alpha\,\Delta\gamma$, $\Delta\beta\,\Delta\gamma$, $\Delta\gamma^2$, in the developments of

$$\left[\delta^2\frac{\delta W}{\delta\alpha}\right],\quad \left[\delta^2\frac{\delta W}{\delta\beta}\right],$$

may be neglected, as being of the third dimension. And if, for further abridgment, we put x, y, instead of Δx, Δy, in the equations (Q^4) to denote the coordinates of the intersection of the near ray with the plane of xy, that is, with the plane of aberration passing through the principal focus, and denote the partial differential coefficients

$$\frac{\delta^3 W}{\delta\alpha^3},\quad \frac{\delta^3 W}{\delta\alpha^2\,\delta\beta},\quad \frac{\delta^3 W}{\delta\alpha\,\delta\beta^2},\quad \frac{\delta^3 W}{\delta\beta^3},$$

by A, B, C, D, we shall have

$$\left.\begin{aligned} x\frac{\delta^2 v}{\delta\alpha^2} + y\frac{\delta^2 v}{\delta\alpha\,\delta\beta} &= \tfrac{1}{2}(Aa^2 + 2Bab + Cb^2),\\ x\frac{\delta^2 v}{\delta\alpha\,\delta\beta} + y\frac{\delta^2 v}{\delta\beta^2} &= \tfrac{1}{2}(Ba^2 + 2Cab + Db^2),\end{aligned}\right\} \tag{S^4}$$

and by elimination,

$$\left.\begin{aligned} 2v''x &= \left(A\frac{\delta^2 v}{\delta\beta^2} - B\frac{\delta^2 v}{\delta\alpha\,\delta\beta}\right)a^2 + 2\left(B\frac{\delta^2 v}{\delta\beta^2} - C\frac{\delta^2 v}{\delta\alpha\,\delta\beta}\right)ab + \left(C\frac{\delta^2 v}{\delta\beta^2} - D\frac{\delta^2 v}{\delta\alpha\,\delta\beta}\right)b^2,\\ 2v''y &= \left(B\frac{\delta^2 v}{\delta\alpha^2} - A\frac{\delta^2 v}{\delta\alpha\,\delta\beta}\right)a^2 + 2\left(C\frac{\delta^2 v}{\delta\alpha^2} - B\frac{\delta^2 v}{\delta\alpha\,\delta\beta}\right)ab + \left(D\frac{\delta^2 v}{\delta\alpha^2} - C\frac{\delta^2 v}{\delta\alpha\,\delta\beta}\right)b^2,\end{aligned}\right\} \tag{T^4}$$

v'' having the same meaning as before.

20. The equations (S^4), or (T^4), express the connexion between the coordinates x, y, of the intersection of a near ray with the plane of aberration passing through the principal focus, and the coordinates a, b, of the intersection of the same near ray with another plane of aberration, parallel to the former, and at a distance from it equal to unity: they serve therefore to resolve the questions that have reference to this connexion. The most interesting questions of this kind, are those which relate to the comparative condensation of the near rays, in crossing the two planes of aberration. Let us therefore consider an infinitely small rectangle $\delta a \,.\, \delta b$ on the plane of a, b, having for coordinates of its four corners,

$$\text{Ist. } a,\ b; \quad \text{IInd. } a + \delta a,\ b; \quad \text{IIId. } a,\ b + \delta b; \quad \text{IVth. } a + \delta a,\ b + \delta b:$$

the rays which pass inside this little rectangle, will, at the plane of xy, be diffused over a little parallelogram, of which the coordinates of the corners are

$$\text{Ist. } x,\ y, \quad \text{IInd. } x + \frac{\delta x}{\delta a}\,\delta a,\ y + \frac{\delta y}{\delta a}\,\delta a; \quad \text{IIId. } x + \frac{\delta x}{\delta b}\,\delta b,\ y + \frac{\delta y}{\delta b}\,\delta b;$$

$$\text{IVth. } x + \frac{\delta x}{\delta a}\,\delta a + \frac{\delta x}{\delta b}\,\delta b,\ y + \frac{\delta y}{\delta a}\,\delta a + \frac{\delta y}{\delta b}\,\delta b:$$

the partial differential coefficients

$$\frac{\delta x}{\delta a}, \quad \frac{\delta x}{\delta b}, \quad \frac{\delta y}{\delta a}, \quad \frac{\delta y}{\delta b},$$

being obtained by differentiating the equations (S^4), or (T^4). The area of the parallelogram on the plane of xy is

$$\pm \left(\frac{\delta x}{\delta a} \frac{\delta y}{\delta b} - \frac{\delta x}{\delta b} \frac{\delta y}{\delta a} \right) \delta a\, \delta b;$$

its ratio to the rectangle $\delta a\, \delta b$, is therefore expressed by

$$\pm \left(\frac{\delta x}{\delta a} \frac{\delta y}{\delta b} - \frac{\delta x}{\delta b} \frac{\delta y}{\delta a} \right):$$

and by the equations (T^4), or (S^4),

$$\left(\frac{\delta x}{\delta a} \frac{\delta y}{\delta b} - \frac{\delta x}{\delta b} \frac{\delta y}{\delta a} \right) = \frac{M''}{v''}, \tag{U^4}$$

if we put

$$M'' = (Aa + Bb)(Ca + Db) - (Ba + Cb)^2. \tag{V^4}$$

The smaller the quantity M'' is, the more will the rays which pass through the little rectangle $\delta a\, \delta b$, be condensed at the principal focus; so that the curves upon the plane of a, b, which have for equation

$$M'' = \text{const.}, \tag{W^4}$$

may be called *lines of uniform condensation*; and we see, by (V^4), that these curves will be ellipses or hyperbolas, according as N'' is positive or negative, if we put for abridgment,

$$4(B^2 - AC)(C^2 - BD) - (AD - BC)^2 = N''. \tag{X^4}$$

These elliptic or hyperbolic curves are all concentric and similar, and their axes are all contained on the same pair of indefinite right lines, which are perpendicular to each other and to the given ray; and the planes which pass through the ray, and through these axes of the line (W^4), will coincide with the planes of xz, yz, if the following condition be satisfied:

$$AD - BC = 0, \tag{Y^4}$$

that is

$$\frac{\delta^3 W}{\delta\alpha^3}\frac{\delta^3 W}{\delta\beta^3} = \frac{\delta^3 W}{\delta\alpha^2\,\delta\beta}\frac{\delta^3 W}{\delta\alpha\,\delta\beta^2}. \tag{Z^4}$$

This condition is independent of the magnitude of the unit of distance, by which we have supposed the two planes of aberration to be separated: there are therefore an infinite number of systems of ellipses or hyperbolas, similar to the system (W^4), and all having their axes contained in the same pair of rectangular planes, which pass through the principal ray: and it is natural to take these planes for the planes of xz, yz, the plane of xy being still the same plane of aberration as before. And thus, the intersections of these three rectangular planes, may be considered as furnishing, in general, *three natural axes of an optical system*, which are perpendicular each to each, and cross in the principal focus. These natural axes possess many other properties, of which we hope to treat hereafter; but in the foregoing remarks we have only aimed to shew, by some selected instances, the possibility of deducing the geometrical properties of optical systems of rays, from the fundamental formula (A),

$$\delta \int v\,ds = \frac{\delta v}{\delta\alpha}\,\delta x + \frac{\delta v}{\delta\beta}\,\delta y + \frac{\delta v}{\delta\gamma}\,\delta z,$$

with the assistance of the characteristic function V, and of the connected function W: and believing that this object has been accomplished, we shall conclude the present Supplement.