ON THE ARGUMENT OF ABEL

By

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NOTE ON THE TEXT

The text of this edition is taken from the 18th volume of the *Transactions of the Royal Irish Academy.* A small number of obvious typographical errors have been corrected without comment in articles 1, 3, 5, 6, 10 and 21.

David R. Wilkins

Dublin, February 2000

On the Argument of ABEL, *respecting the Impossibility of expressing a Root of any General Equation above the Fourth Degree, by any finite Combination of Radicals and Rational Functions.*

By Sir WILLIAM ROWAN HAMILTON.

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[1.] Let $a_1, a_2, \dots a_n$ be any n arbitrary quantities, or independent variables, real or imaginary, and let $a'_1, a'_2, \dots a'_{n'}$ be any n' radicals, such that

$$a_1'^{\alpha'_1} = f_1(a_1, \dots a_n), \quad \dots \quad a_{n'}'^{\alpha'_{n'}} = f_{n'}(a_1, \dots a_n);$$

again, let $a''_1, \dots a''_{n''}$ be n'' new radicals, such that

$$\begin{aligned} a_1''^{\alpha''_1} &= f'_1(a'_1, \dots a'_{n'}, a_1, \dots a_n), \\ &\dots\dots\dots \\ a_{n''}''^{\alpha''_{n''}} &= f'_{n''}(a'_1, \dots a'_{n'}, a_1, \dots a_n); \end{aligned}$$

and so on, till we arrive at a system of equations of the form

$$\begin{aligned} a_1^{(m)\alpha_1^{(m)}} &= f_1^{(m-1)}(a_1^{(m-1)}, \dots a_{n^{(m-1)}}^{(m-1)}, a_1^{(m-2)}, \dots a_{n^{(m-2)}}^{(m-2)}, \dots a_1, \dots a_n), \\ &\dots\dots\dots \\ a_{n^{(m)}}^{(m)\alpha_{n^{(m)}}^{(m)}} &= f_{n^{(m)}}^{(m-1)}(a_1^{(m-1)}, \dots a_{n^{(m-1)}}^{(m-1)}, a_1^{(m-2)}, \dots a_{n^{(m-2)}}^{(m-2)}, \dots a_1, \dots a_n), \end{aligned}$$

the exponents $\alpha_i^{(k)}$ being all integral and prime numbers greater than unity, and the functions $f_i^{(k-1)}$ being rational, but all being otherwise arbitrary. Then, if we represent by $b^{(m)}$ any rational function $f^{(m)}$ of all the foregoing quantities $a_i^{(k)}$,

$$b^{(m)} = f^{(m)}(a_1^{(m)}, \dots a_{n^{(m)}}^{(m)}, a_1^{(m-1)}, \dots a_{n^{(m-1)}}^{(m-1)}, \dots a_1, \dots a_n),$$

we may consider the quantity $b^{(m)}$ as being also an irrational function of the n original quantities, $a_1, \dots a_n$; in which latter view it may be said, according to a phraseology proposed by ABEL, to be an *irrational function of the* m^{th} *order*: and may be regarded as the general type of every conceivable function of any finite number of independent variables, which can be formed by any finite number of additions, subtractions, multiplications, divisions, elevations to powers, and extraction of roots of functions; since it is obvious that any extraction of a radical with a composite exponent, such as $\sqrt[\alpha'_2\alpha'_1]{f_1}$, may be reduced to a system of successive extractions of radicals with prime exponents, such as

$$\sqrt[\alpha'_1]{f_1} = f'_1, \quad \sqrt[\alpha'_2]{f'_1} = f''_1.$$

Insomuch that the question, "Whether it be possible to express a root x of the general equation of the n^{th} degree,

$$x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n = 0,$$

in terms of the coefficients of that equation, by any finite combination of radicals and rational functions?", is, as Abel has remarked, equivalent to the question, "Whether it be possible to equate a root of the general equation of any given degree to an irrational function of the coefficients of that equation, which function shall be of any finite order m?" or to this other question: "Is it possible to satisfy, by any function of the form $b^{(m)}$, the equation

$$b^{(m)n} + a_1 b^{(m)n-1} + \cdots + a_{n-1} b^{(m)} + a_n = 0,$$

in which the exponent n is given, but the coefficients $a_1, a_2, \ldots\ a_n$ are arbitrary?"

[2.] For the cases $n = 2$, $n = 3$, $n = 4$, this question has long since been determined in the affirmative, by the discovery of the known solutions of the general quadratic, cubic, and biquadratic equations.

Thus, for $n = 2$, it has long been known that a root x of the general quadratic equation,

$$x^2 + a_1 x + a_2 = 0,$$

can be expressed as a finite irrational function of the two arbitrary coefficients a_1, a_2, namely, as the following function, which is of the first order:

$$x = b' = f'(a'_1, a_1, a_2) = \frac{-a_1}{2} + a'_1,$$

the radical a'_1 being such that

$$a'^2_1 = f_1(a_1, a_2) = \frac{a_1^2}{4} - a_2;$$

insomuch that, with this form of the irrational function b', the equation

$$b'^2 + a_1 b' + a_2 = 0$$

is satisfied, independently of the quantities a_1 and a_2, which remain altogether arbitrary.

Again, it is well known that for $n = 3$, that is, in the case of the general cubic equation

$$x^3 + a_1 x^2 + a_2 x + a_3 = 0,$$

a root x may be expressed as an irrational function of the three arbitrary coefficients a_1, a_2, a_3, namely as the following function, which is of the second order:

$$\begin{aligned} x = b'' &= f''(a_1'', a_1', a_1, a_2, a_3) \\ &= -\frac{a_1}{3} + a_1'' + \frac{c_2}{a_1''}; \end{aligned}$$

the radical of highest order, a_1'', being defined by the equation

$$\begin{aligned} a_1''^3 &= f_1'(a_1', a_1, a_2, a_3) \\ &= c_1 + a_1', \end{aligned}$$

and the subordinate radical a_1' being defined by this other equation

$$a_1'^2 = f_1(a_1, a_2, a_3) = c_1^2 - c_2^3,$$

while c_1 and c_2 denote for abridgment the following two rational functions:

$$\begin{aligned} c_1 &= -\tfrac{1}{54}(2a_1^3 - 9a_1a_2 + 27a_3), \\ c_2 &= \tfrac{1}{9}(a_1^2 - 3a_2); \end{aligned}$$

so that, with this form of the irrational function b'', the equation

$$b''^3 + a_1b''^2 + a_2b'' + a_3 = 0$$

is satisfied, without any restriction being imposed on the three coefficients a_1, a_2, a_3.

For $n = 4$, that is, for the case of the general biquadratic equation

$$x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0,$$

it is known in like manner, that a root can be expressed as a finite irrational function of the coefficients, namely as the following function, which is of the third order:

$$x = b''' = f'''(a_1''', a_2''', a_1'', a_1', a_1, a_2, a_3, a_4) = -\frac{a_1}{4} + a_1''' + a_2''' + \frac{e_4}{a_1'''a_2'''};$$

wherein

$$\begin{aligned} a_1'''^2 &= f_1''(a_1'', a_1', a_1, a_2, a_3, a_4) = e_3 + a_1'' + \frac{e_2}{a_1''}, \\ a_2'''^2 &= f_2''(a_1'', a_1', a_1, a_2, a_3, a_4) = e_3 + \rho_3 a_1'' + \frac{e_2}{\rho_3 a_1''}, \\ a_1''^3 &= f_1'(a_1', a_1, a_2, a_3, a_4) = e_1 + a_1', \\ a_1'^2 &= f_1(a_1, a_2, a_3, a_4) = e_1^2 - e_2^3; \end{aligned}$$

e_4, e_3, e_2, e_1 denoting for abridgment the following rational functions:

$$\begin{aligned}
e_4 &= \tfrac{1}{64}(-a_1^3 + 4a_1a_2 - 8a_3),\\
e_3 &= \tfrac{1}{48}(3a_1^2 - 8a_2),\\
e_2 &= \tfrac{1}{144}(-3a_1a_3 + a_2^2 + 12a_4),\\
e_1 &= \tfrac{1}{2}(3e_2e_3 - e_3^3 + e_4^2)\\
&= \tfrac{1}{3456}(27a_1^2a_4 - 9a_1a_2a_3 + 2a_2^3 - 72a_2a_4 + 27a_3^2),
\end{aligned}$$

and ρ_3 being a root of the numerical equation

$$\rho_3^2 + \rho_3 + 1 = 0.$$

It is known also, that a root x of the same general biquadratic equation may be expressed in another way, as an irrational function of the fourth order of the same arbitrary coefficients a_1, a_2, a_3, a_4, namely the following:

$$\begin{aligned}
x = b^{\mathrm{IV}} &= f^{\mathrm{IV}}(a_1^{IV}, a_1''', a_1'', a_1', a_1, a_2, a_3, a_4)\\
&= -\frac{a_1}{4} + a_1''' + a_1^{\mathrm{IV}};
\end{aligned}$$

the radical a_1^{IV} being defined by the equation

$$a_1^{\mathrm{IV}2} = f_1'''(a_1''', a_1'', a_1', a_1, a_2, a_3, a_4) = -a_1'''^2 + 3e_3 + \frac{2e_4}{a_1'''},$$

while a_1''', a_1'', a_1', and e_4, e_3, e_2, e_1, retain their recent meanings. Insomuch that either the function of third order b''', or the function of fourth order b^{IV}, may be substituted for x in the general biquadratic equation; or, to express the same thing otherwise, the two equations following:

$$b'''^4 + a_1b'''^3 + a_2b'''^2 + a_3b''' + a_4 = 0,$$

and

$$b^{\mathrm{IV}4} + a_1b^{\mathrm{IV}3} + a_2b^{\mathrm{IV}2} + a_3b^{\mathrm{IV}} + a_4 = 0,$$

are both identically true, in virtue merely of the forms of the irrational functions b''' and b^{IV}, and independently of the values of the four arbitrary coefficients a_1, a_2, a_3, a_4.

But for higher values of n the question becomes more difficult; and even for the case $n = 5$, that is, for the general equation of the fifth degree,

$$x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5 = 0,$$

the opinions of mathematicians appear to be not yet entirely agreed respecting the possibility or impossibility of expressing a root as a function of the coefficients by any finite combination of radicals and rational functions: or, in other words, respecting the possibility or impossibilty of satisfying, by any irrational function $b^{(m)}$ of any finite order, the equation

$$b^{(m)5} + a_1b^{(m)4} + a_2b^{(m)3} + a_3b^{(m)2} + a_4b^{(m)} + a_5 = 0,$$

the five coefficients a_1, a_2, a_3, a_4, a_5, remaining altogether arbitrary. To assist in deciding opinions upon this important question, by developing and illustrating (with alterations) the admirable argument of ABEL against the possibility of any such expression for a root of the general equation of the fifth, or any higher degree; and by applying the principles of the same argument, to show that no expression of the same kind exists for any root of any general but lower equation, (quadratic, cubic, or biquadratic,) essentially distinct from those which have long been known; is the chief object of the present paper.

[3.] In general, if we call an irrational function *irreducible*, when it is impossible to express that function, or any one of its component radicals, by any smaller number of extractions of prime roots of variables, than the number which the actual expression of that function or radical involves; even by introducing roots of constant quantities, or of numerical equations, which roots are in this whole discussion considered as being themselves constant quantities, so that they neither influence the order of an irrational function, nor are included among the radicals denoted by the symbols a'_1, &c.; then it is not difficult to prove that such *irreducible irrational functions* possess several properties in common, which are adapted to assist in deciding the question just now stated.

In the first place it may be observed, that, by an easy preparation, the general irrational function $b^{(m)}$ of any order m may be put under the form

$$b^{(m)} = \sum_{\beta_i^{(m)}<\alpha_i^{(m)}} .(b^{(m-1)}_{\beta_1^{(m)},\dots\beta_{n^{(m)}}^{(m)}} .\, a_1^{(m)\beta_1^{(m)}} \dots a_{n^{(m)}}^{(m)\beta_{n^{(m)}}^{(m)}}),$$

in which the coefficient $b^{(m-1)}_{\beta_1^{(m)},\dots\beta_{n^{(m)}}^{(m)}}$ is a function of the order $m-1$, or of a lower order; the exponent $\beta_i^{(m)}$ is zero, or any positive integer less than the prime number $\alpha_i^{(m)}$ which enters as exponent into the equation of definition of the radical $a_i^{(m)}$, namely,

$$a_i^{(m)\alpha_i^{(m)}} = f_i^{(m-1)};$$

and the sign of summation extends to all the $\alpha_1^{(m)}.\alpha_2^{(m)}\dots\alpha_{n^{(m)}}^{(m)}$ terms which have exponents $\beta_i^{(m)}$ subject to the condition just now mentioned.

For, inasmuch as $b^{(m)}$ is, by supposition, a rational function $f^{(m)}$ of all the radicals $a_i^{(k)}$, it is, with respect to any radical of highest order, such as $a_i^{(m)}$, a function of the form

$$b^{(m)} = \frac{\mathrm{N}(a_i^{(m)})}{\mathrm{M}(a_i^{(m)})},$$

M and N being here used as signs of some *whole* functions, or finite integral polynomes. Now, if we denote by ρ_α any root of the numerical equation

$$\rho_\alpha^{(\alpha-1)} + \rho_\alpha^{(\alpha-2)} + \rho_\alpha^{(\alpha-3)} + \dots + \rho_\alpha^2 + \rho_\alpha + 1 = 0,$$

so that ρ_α is at the same time a root of unity, because the last equation gives

$$\rho_\alpha^\alpha = 1;$$

and if we suppose the number α to be prime, so that

$$\rho_\alpha, \rho_\alpha^2, \rho_\alpha^3, \dots \rho_\alpha^{(\alpha-1)}$$

are, in some arrangement or other, the $\alpha - 1$ roots of the equation above assigned: then, the product of all the $\alpha - 1$ whole functions following,

$$\mathrm{M}(\rho_\alpha a) \,.\, \mathrm{M}(\rho_\alpha^2 a) \dots \mathrm{M}(\rho_\alpha^{(\alpha-1)} a) = \mathrm{L}(a),$$

is not only itself a whole function of a, but is one which, when multiplied by $\mathrm{M}(a)$, gives a product of the form

$$\mathrm{L}(a) \,.\, \mathrm{M}(a) = \mathrm{K}(a^\alpha),$$

K being here (as well as L) a sign of some whole function. If then we form the product

$$\mathrm{M}(\rho_{\alpha_i^{(m)}} a_i^{(m)}) \,.\, \mathrm{M}(\rho^2_{\alpha_i^{(m)}} a_i^{(m)}) \dots \mathrm{M}(\rho_{\alpha_i^{(m)}}^{\alpha_i^{(m)}-1} a_i^{(m)}) = \mathrm{L}(a_i^{(m)}),$$

and multiply, by it, both numerator and denominator of the recently assigned expression for $b^{(m)}$, we obtain this new expression for that general irrational function,

$$b^{(m)} = \frac{\mathrm{L}(a_i^{(m)}) \,.\, \mathrm{N}(a_i^{(m)})}{\mathrm{L}(a_i^{(m)}) \,.\, \mathrm{M}(a_i^{(m)})} = \frac{\mathrm{L}(a_i^{(m)}) \,.\, \mathrm{N}(a_i^{(m)})}{\mathrm{K}(a_i^{(m)\alpha_i^{(m)}})} = \frac{\mathrm{L}(a_i^{(m)}) \,.\, \mathrm{N}(a_i^{(m)})}{\mathrm{K}(f_i^{(m-1)})} = \mathrm{I}(a_i^{(m)});$$

the characteristic I denoting here some function, which, relatively to the radical $a_i^{(m)}$, is whole, so that it may be thus developed,

$$b^{(m)} = \mathrm{I}(a_i^{(m)}) = \mathrm{I}_0 + \mathrm{I}_1 a_i^{(m)} + \mathrm{I}_2 a_i^{(m)2} + \dots + \mathrm{I}_r \alpha_i^{(m)r},$$

r being a finite positive integer, and the coefficients $\mathrm{I}_0, \mathrm{I}_1, \dots \mathrm{I}_r$ being, in general, functions of the m^{th} order, but not involving the radical $a_i^{(m)}$. And because the definition of that radical gives

$$a_i^{(m)h} = a_i^{(m)g} \,.\, (f_i^{(m-1)})^e,$$

if

$$h = g + e\alpha_i^{(m)},$$

it is unnecessary to retain in evidence any of its powers of which the exponents are not less than $\alpha_i^{(m)}$; we may therefore put the development of $b^{(m)}$ under the form

$$b^{(m)} = \mathrm{H}_0 + \mathrm{H}_1 a_i^{(m)} + \dots + \mathrm{H}_{\alpha_i^{(m)}-1} (a_i^{(m)})^{\alpha_i^{(m)}-1},$$

the coefficients $\text{H}_0, \text{H}_1, \ldots$ being still, in general, functions of the m^{th} order, not involving the radical $a_i^{(m)}$. It is clear that by a repetition of this process of transformation, the radicals $a_1^{(m)}, \ldots a_{n^{(m)}}^{(m)}$ may all be removed from the denominator of the rational function $f^{(m)}$; and that their exponents in the transformed numerator may all be depressed below the exponents which define those radicals: by which means, the development above announced for the general irrational function $b^{(m)}$ may be obtained; wherein the coefficient $b^{(m-1)}_{\beta_1^{(m)},\ldots\beta^{(m)}_{n^{(m)}}}$ admits of being analogously developed.

For example, the function of the second order,

$$b'' = -\frac{a_1}{3} + a_1'' + \frac{c_2}{a_1''},$$

which was above assigned as an expression for a root of the general cubic equation, may be developed thus:

$$b'' = \sum_{\beta_1''<3} .(b'_{\beta_1''} \,.\, a_1''^{\beta_1''}) = b'_0 + b'_1 a_1'' + b'_2 a_1''^2;$$

in which

$$b'_0 = -\frac{a_1}{3}, \quad b'_1 = 1, \quad b'_2 = \frac{c_2}{a_1''^3} = \frac{c_2}{f'_1} = \frac{c_2}{c_1 + a'_1}.$$

And this last coefficient b'_2, which is itself a function of the first order, may be developed thus:

$$b'_2 = \frac{c_2}{c_1 + a'_1} = \text{B}' = \sum_{\beta'_1<2} .(\text{B}_{\beta'_1} \,.\, a_1'^{\beta'_1}) = \text{B}_0 + \text{B}_1 a'_1;$$

in which

$$\text{B}_0 = \frac{c_2 c_1}{c_1^2 - a_1'^2} = \frac{c_2 c_1}{c_1^2 - f_1} = \frac{c_2 c_1}{c_2^3} = \frac{c_1}{c_2^2}, \quad \text{B}_1 = \frac{-1}{c_2^2}.$$

Again, the function of the third order,

$$b''' = \frac{-a_1}{4} + a_1''' + a_2''' + \frac{e_4}{a_1''' a_2'''},$$

which expresses a root of the general biquadratic equation, may be developed as follows:

$$\begin{aligned} b''' &= \sum_{\substack{\beta_1'''<2 \\ \beta_2'''<2}} .(b''_{\beta_1''',\beta_2'''} \,.\, a_1'''^{\beta_1'''} \,.\, a_2'''^{\beta_2'''}) \\ &= b''_{0,0} + b''_{1,0} a_1''' + b''_{0,1} a_2''' + b''_{1,1} a_1''' a_2'''; \end{aligned}$$

in which

$$b''_{0,0} = \frac{-a_1}{4}, \quad b''_{1,0} = 1, \quad b''_{0,1} = 1,$$

and

$$\begin{aligned} b''_{1,1} &= \frac{e_4}{a_1'''^2 \,.\, a_2'''^2} = \frac{e_4}{f_1'' \,.\, f_2''} = \frac{e_4}{\left(e_3 + a_1'' + \dfrac{e_2}{a_1''}\right)\left(e_3 + \rho_3 a_1'' + \dfrac{e_2}{\rho_3 a_1''}\right)} \\ &= \frac{1}{e_4}\left(e_3 + \rho_3^2 a_1'' + \frac{e_2}{\rho_3^2 a_1''}\right). \end{aligned}$$

And this last coefficient $b''_{1,1}$, which is itself a function of the second order, may be developed thus:

$$b''_{1,1} = \mathrm{B}'' = \sum_{\beta''_1 < 3} .(\mathrm{B}'_{\beta''_1} \,.\, a_1''^{\beta''_1}) = \mathrm{B}'_0 + \mathrm{B}'_1 a''_1 + \mathrm{B}'_2 a_1''^2;$$

in which

$$\mathrm{B}'_0 = \frac{e_3}{e_4}, \quad \mathrm{B}'_1 = \frac{\rho_3^2}{e_4}, \quad \mathrm{B}'_2 = \frac{\rho_3 e_2}{e_4 a_1''^3} = \frac{\rho_3 e_2}{e_4(e_1 + a'_1)} = \frac{\rho_3(e_1 - a'_1)}{e_4 e_2^2}.$$

So that, upon the whole, these functions b'' and b''', which express, respectively, roots of the general cubic and biquadratic equations, may be put under the following forms, whch involve no radicals in denominators:

$$b'' = \frac{-a_1}{3} + a''_1 + (c_1 - a'_1)\left(\frac{a''_1}{c_2}\right)^2,$$

and

$$b'' = \frac{-a_1}{4} + a'''_1 + a'''_2 + \frac{1}{e_4}\left\{e_3 + \rho_3^2 a''_1 + \rho_3(e_1 - a'_1)\left(\frac{a''_1}{e_2}\right)^2\right\} a'''_1 a'''_2;$$

and the functions f''_1, f''_2, which enter into the equations of definition of the radicals a'''_1, a'''_2, namely into the equations

$$a_1'''^2 = f''_1, \quad a_2'''^2 = f''_2,$$

may in like manner be expressed so as to involve no radicals in denominators, namely thus:

$$a_1'''^2 = e_3 + a''_1 + (e_1 - a'_1)\left(\frac{a''_1}{e_2}\right)^2,$$

$$a_2'''^2 = e_3 + \rho_3 a''_1 + \rho_3^2(e_1 - a'_1)\left(\frac{a''_1}{e_2}\right)^2.$$

It would be easy to give other instances of the same sort of transformation, but it seems unnecessary to do so.

[4.] It is important in the next place to observe, that any term of the foregoing general development of the general irrational function $b^{(m)}$, may be isolated from the rest, and expressed separately, as follows. Let $b^{(m)}_{\gamma_1^{(m)},\ldots\gamma_{n^{(m)}}^{(m)}}$ denote a new irrational function, which is formed from $b^{(m)}$ by changing every radical such as $a_i^{(m)}$ to a corresponding product such as $\rho_{\alpha_i^{(m)}}^{\gamma_i^{(m)}} a_i^{(m)}$, in which $\rho_{\alpha_i^{(m)}}$ is, as before, a root of unity; so that

$$b^{(m)}_{\gamma_1^{(m)},\ldots\gamma_{n^{(m)}}^{(m)}} = \sum_{\beta_i^{(m)} < \alpha_i^{(m)}} .(b^{(m-1)}_{\beta_1^{(m)},\ldots\beta_{n^{(m)}}^{(m)}} \,.\, \rho_{\alpha_1^{(m)}}^{\beta_1^{(m)}\gamma_1^{(m)}} \ldots \rho_{\alpha_{n^{(m)}}^{(m)}}^{\beta_{n^{(m)}}^{(m)}\gamma_{n^{(m)}}^{(m)}} \,.\, a_1^{(m)\beta_1^{(m)}} \ldots a_{n^{(m)}}^{(m)\beta_{n^{(m)}}^{(m)}});$$

and let any isolated term of the corresponding development of $b^{(m)}$ or $b^{(m)}_{0,\ldots 0}$ be denoted by the symbol

$$t^{(m)}_{\beta_1^{(m)},\ldots\beta_{n^{(m)}}^{(m)}} = b^{(m-1)}_{\beta_1^{(m)},\ldots\beta_{n^{(m)}}^{(m)}} \,.\, a_1^{(m)\beta_1^{(m)}} \ldots a_{n^{(m)}}^{(m)\beta_{n^{(m)}}^{(m)}};$$

we shall then have, as the announced expression for the isolated term, the following:

$$t^{(m)}_{\beta_1^{(m)},\dots\beta^{(m)}_{n^{(m)}}} = \frac{1}{\alpha_1^{(m)}\dots\alpha^{(m)}_{n^{(m)}}} \cdot \sum_{\gamma_i^{(m)}<\alpha_i^{(m)}} .(b^{(m)}_{\gamma_1^{(m)},\dots\gamma^{(m)}_{n^{(m)}}} \cdot \rho_{\alpha_1^{(m)}}^{-\beta_1^{(m)}\gamma_1^{(m)}} \dots \rho_{\alpha^{(m)}_{n^{(m)}}}^{-\beta^{(m)}_{n^{(m)}}\gamma^{(m)}_{n^{(m)}}});$$

the sign of summation here extending to all those terms in which every index such as $\gamma_1^{(m)}$ is equal to zero or to some positive integer less than $\alpha_i^{(m)}$.

Thus, in the case of the function of second order b'', which represents, as we have seen, a root of the general cubic equation, if we wish to obtain an isolated expression for any term $t''_{\beta_1''}$ of its development already found, namely the development

$$b'' = \sum_{\beta_1''<3} .(b'_{\beta_1''} \cdot a_1''^{\beta_1''}) = b_0' + b_1'a_1'' + b_2'a_1''^2 = t_0'' + t_1'' + t_2'',$$

we have only to introduce the function

$$b''_{\gamma_1''} = \sum_{\beta_1''<3} .(b'_{\beta_1''} \cdot \rho_3^{\beta_1''\gamma_1''} \cdot a_1''^{\beta_1''}) = b_0' + b_1'\rho_3^{\gamma_1''} a_1'' + b_2'\rho_3^{2\gamma_1''} a_1''^2,$$

and to employ the formula

$$t''_{\beta_1''} = b'_{\beta_1''} \cdot a_1''^{\beta_1''} = \tfrac{1}{3} \cdot \sum_{\gamma_1''<3} .(b''_{\gamma_1''} \cdot \rho_3^{-\beta_1''\gamma_1''}) = \tfrac{1}{3}(b_0'' + \rho_3^{-\beta_1''} b_1'' + \rho_3^{-2\beta_1''} b_2'').$$

In particular,

$$\begin{aligned} t_0'' &= b_0' = \tfrac{1}{3}(b_0'' + b_1'' + b_2''),\\ t_1'' &= b_1'a_1'' = \tfrac{1}{3}(b_0'' + \rho_3^{-1}b_1'' + \rho_3^{-2}b_2''),\\ t_2'' &= b_2'a_1''^2 = \tfrac{1}{3}(b_0'' + \rho_3^{-2}b_1'' + \rho_3^{-4}b_2''), \end{aligned}$$

in which

$$\begin{aligned} b_0'' &= b_0' + b_1'a_1'' + b_2'a_1''^2 \;(= b''),\\ b_1'' &= b_0' + b_1'\rho_3 a_1'' + b_2'\rho_3^2 a_1''^2,\\ b_2'' &= b_0' + b_1'\rho_3^2 a_1'' + b_2'\rho_3^4 a_1''^2, \end{aligned}$$

and in which it is to be remembered that

$$\rho_3^2 + \rho_3 + 1 = 0, \quad \text{and therefore} \quad \rho_3^3 = 1.$$

Again, if we wish to isolate any term $t_{\beta_1''',\beta_2'''}$ of the development above assigned for the function of third order b''', which represents a root of the general biquadratic equation, we may employ the formula

$$\begin{aligned} t'''_{\beta_1''',\beta_2'''} = b''_{\beta_1''',\beta_2'''} \cdot a_1'''^{\beta_1'''} \cdot a_2'''^{\beta_2'''} &= \frac{1}{2.2} \cdot \sum_{\substack{\gamma_1'''<2\\ \gamma_2'''<2}} .(b'''_{\gamma_1''',\gamma_2'''} \cdot \rho_2^{-\beta_1'''\gamma_1'''} \cdot \rho_2^{-\beta_2'''\gamma_2'''})\\ &= \tfrac{1}{4}\{b'''_{0,0} + (-1)^{-\beta_1'''} b'''_{1,0} + (-1)^{-\beta_2'''} b'''_{0,1} + (-1)^{-(\beta_1'''+\beta_2''')} b'''_{1,1}\}; \end{aligned}$$

in which we have introduced the function

$$b'''_{\gamma'''_1,\gamma'''_2} = \sum_{\substack{\beta'''_1<2\\ \beta'''_2<2}} .(b''_{\beta'''_1,\beta'''_2} \,.\, \rho_2^{\beta'''_1\gamma'''_1} \,.\, \rho_2^{\beta'''_2\gamma'''_2} \,.\, a_1^{\prime\prime\prime\beta'''_1} \,.\, a_2^{\prime\prime\prime\beta'''_2})$$

$$= b''_{0,0} + (-1)^{\gamma'''_1} b''_{1,0}a'''_1 + (-1)^{\gamma'''_2} b''_{0,1}a'''_2 + (-1)^{\gamma'''_1+\gamma'''_2} b''_{1,1}a'''_1a'''_2;$$

so that, in particular, we have the four expressions

$$\begin{aligned}
t'''_{0,0} &= b''_{0,0} = \tfrac{1}{4}(b'''_{0,0} + b'''_{1,0} + b'''_{0,1} + b'''_{1,1}),\\
t'''_{1,0} &= b''_{1,0}a'''_1 = \tfrac{1}{4}(b'''_{0,0} - b'''_{1,0} + b'''_{0,1} - b'''_{1,1}),\\
t'''_{0,1} &= b''_{0,1}a'''_2 = \tfrac{1}{4}(b'''_{0,0} + b'''_{1,0} - b'''_{0,1} - b'''_{1,1}),\\
t'''_{1,1} &= b''_{1,1}a'''_1a'''_2 = \tfrac{1}{4}(b'''_{0,0} - b'''_{1,0} - b'''_{0,1} + b'''_{1,1}),
\end{aligned}$$

in which

$$\begin{aligned}
b'''_{0,0} &= b''_{0,0} + b''_{1,0}a'''_1 + b''_{0,1}a'''_2 + b''_{1,1}a'''_1a'''_2,\\
b'''_{1,0} &= b''_{0,0} - b''_{1,0}a'''_1 + b''_{0,1}a'''_2 - b''_{1,1}a'''_1a'''_2,\\
b'''_{0,1} &= b''_{0,0} + b''_{1,0}a'''_1 - b''_{0,1}a'''_2 - b''_{1,1}a'''_1a'''_2,\\
b'''_{1,1} &= b''_{0,0} - b''_{1,0}a'''_1 - b''_{0,1}a'''_2 + b''_{1,1}a'''_1a'''_2.
\end{aligned}$$

In these examples, the truth of the results is obvious; and the general demonstration follows easily from the properties of the roots of unity.

[5.] We have hitherto made no use of the assumed *irreducibility* of the irrational function $b^{(m)}$. But taking now this property into account, we soon perceive that the component radicals $a_i^{(k)}$, which enter into the composition of this irreducible function, must not be subject to, nor even compatible with, any equations or equation of condition whatever, except only the equations of definition, which determine those radicals $a_i^{(k)}$, by determining their prime powers $a_i^{(k)\alpha_i^{(k)}}$. For the existence or possibility of any such equation of condition in conjunction with those equations of definition, would enable us to express at least one of the above mentioned radicals as a rational function of others of the same system, and of orders not higher than its own, or even, perhaps, as a rational function of the original variables $a_1, \dots a_n$, though multiplied in general by a root of a numerical equation; and therefore would enable us to diminish the number of extractions of prime roots of functions, which would be inconsistent with the irreducibility supposed.

If fact, if any such equation of condition, involving any radical or radicals of order k, but none of any higher order, were compatible with the equations of definition; then, by some obvious preparations, such as bringing the equation of condition to the form of zero equated to some finite polynomial function of some radical $a_i^{(k)}$ of the k^{th} order; and rejecting, by the methods of equal roots and of the greatest common measure, all factors of this polynome, except those which are unequal among themselves, and are included among the factors of that other polynome which is equated to zero in the corresponding form of the equation of definition of the radical $a_i^{(k)}$; we should find that this last equation of definition

$$a_i^{(k)\alpha_i^{(k)}} - f_i^{(k-1)} = 0$$

must be divisible, either identically, or at least for some suitable system of values of the remaining radicals, by an equation of condition of the form

$$a_i^{(k)g} + \mathrm{G}_1^{(k)} a_i^{(k)g-1} + \cdots + \mathrm{G}_{g-1}^{(k)} a_i^{(k)} + \mathrm{G}_g^{(k)} = 0;$$

g being less than $\alpha_i^{(k)}$, and the coefficients $\mathrm{G}_1^{(k)}, \ldots \mathrm{G}_g^{(k)}$ being functions of orders not higher than k, and not involving the radical $a_i^{(k)}$. Now, if we were to suppose that, for any system of values of the remaining radicals, the coefficients $\mathrm{G}_1^{(k)}, \ldots$ should all be $= 0$, or indeed if even the last of those coefficients should thus vanish, we should then have a new equation of condition, namely the following:

$$f_i^{(k-1)} = 0,$$

which would be obliged to be compatible with the equations of definition of the remaining radicals, and would therefore either conduct at last, by a repetition of the same analysis, to a radical essentially vanishing, and consequently superfluous, among those which have been supposed to enter into the composition of the function $b^{(m)}$; or else would bring us back to the divisibility of an equation of definition by an equation of condition, of the form just now assigned, and with coefficients $\mathrm{G}_1^{(k)}, \ldots \mathrm{G}_g^{(k)}$ which would not all be $= 0$. But for this purpose it would be necessary that a relation, or system of relations, should exist (or at least should be compatible with the remaining equations of definition,) of the form

$$\mathrm{G}_{g-e}^{(k)} = \nu_e a_i^{(k)e},$$

e being less than $\alpha_i^{(k)}$, and ν_e being different from zero, and being a root of a numerical equation; and because $\alpha_i^{(k)}$ is prime, we could find integer numbers λ and μ, which would satisfy the condition

$$\lambda \alpha_i^{(k)} - \mu e = 1;$$

so that, finally, we should have an expression for the radical $a_i^{(k)}$, as a rational function of others of the same system, and of orders not higher than its own, though multiplied in general (as was above announced) by a root of a numerical equation; namely the following expression:

$$a_i^{(k)} = \nu_e^{\mu} \mathrm{G}_{g-e}^{(k)-\mu} f_i^{(k-1)\lambda}.$$

And if we should suppose this last equation to be not identically true, but only to hold good for some systems of values of the remaining radicals, of orders not higher than k, we should still obtain, at least, an equation of condition between those remaining radicals, by raising the expression just found for $a_i^{(k)}$ to the power $\alpha_i^{(k)}$; namely the following equation of condition,

$$f_i^{(k-1)} - (\nu_e^{\mu} \mathrm{G}_{g-e}^{(k)-\mu} f_i^{(k-1)\lambda})^{\alpha_i^{(k)}} = 0,$$

which might then be treated like the former, till at last an expression should be obtained, of the kind above announced, for at least one of the remaining radicals. In every case, therefore, we should be conducted to a diminution of the number of prime roots of variables in the expression of the function $b^{(m)}$, which consequently would not be irreducible.

For example, if an irrational function of the m^{th} order contain any radical $a_i^{(m)}$ of the cubic form, its exponent $\alpha_i^{(m)}$ of the cubic form, its exponent $\alpha_i^{(m)}$ being $= 3$, and its equation of definition being of the form

$$\alpha^{(m)3} = f_i^{(m-1)}(a_1^{(m-1)}, \dots a_{n^{(m-1)}}^{(m-1)}, \dots a_1, \dots a_n);$$

if also the other equations of definition permit us to suppose that this radical *may* be equal to some rational function of the rest, so that an equation of the form

$$a_i^{(m)} + \mathrm{G}_1^{(m)} = 0,$$

(in which the function $\mathrm{G}_1^{(m)}$ does not contain the radical $a_i^{(m)}$,) is *compatible* with the equation of definition

$$a_i^{(m)3} - f_i^{(m-1)} = 0;$$

then, from the forms of these two last mentioned equations, the latter must be *divisible* by the former, at least for some suitable system of values of the remaining radicals: and therefore the following relation, which does not involve the radical $a_i^{(m)}$, namely,

$$f_i^{(m-1)} + \mathrm{G}_1^{(m)3} = 0,$$

must be either identically true, in which case we may substitute for the radical $a_i^{(m)}$, in the proposed function of the m^{th} order, the expression

$$a_i^{(m)} = -\sqrt[3]{1} \,.\, \mathrm{G}_1^{(m)};$$

or at least it must be true as an equation of condition between the remaining radicals, and liable as such to a similar treatment, conducting to an analogous result.

A more simple and specific example is supplied by the following function of the second order,

$$x = -\frac{a_1}{3} + \sqrt[3]{\left(c_1 + \sqrt{c_1^2 - c_2^3}\right)} + \sqrt[3]{\left(c_1 - \sqrt{c_1^2 - c_2^3}\right)},$$

which is not uncommonly proposed as an expression for a root x of the general cubic equation

$$x^3 + a_1 x^2 + a_2 x + a_3 = 0,$$

c_1 and c_2 being certain rational functions of a_1, a_2, a_3, which were assigned in a former article, and which are such that the cubic equation may be thus written:

$$\left(x + \frac{a_1}{3}\right)^3 - 3c_2\left(x + \frac{a_1}{3}\right) - 2c_1 = 0.$$

Putting this function of the second order under the form

$$x = -\frac{a_1}{3} + a_1'' + a_2'',$$

in which the radicals are defined as follows,

$$a_1''^3 = c_1 + a_1', \quad a_2''^3 = c_1 - a_1', \quad a_1'^2 = c_1^2 - c_2^3,$$

we easily perceive that it is *permitted* by these definitions to suppose that the radicals a_1'', a_2'' are connected so as to satisfy the following equation of condition,

$$a_1'' a_2'' = c_2;$$

and even that this supposition *must* be made, in order to render the proposed function of the second order a root of the cubic equation. But the mere knowledge of the *compatibility* of the equation of condition

$$a_2'' - \frac{c_2}{a_1''} = 0$$

with the equation of definition

$$a_2''^3 - (c_1 - a_1') = 0,$$

is sufficient to enable us to infer, from the forms of these two equations, that the latter is divisible by the former, at least for some suitable system of values of the remaining radicals a_1'' and a_1', consistent with their equations of definition; and therefore that the following relation

$$c_1 - a_1' - \left(\frac{c_2}{a_1''}\right)^3 = 0,$$

and the expression

$$a_2'' = \sqrt[3]{1} \, . \, \frac{c_2}{a_1''},$$

are at least consistent with those equations. In the present example, the relation thus arrived at is found to be identically true, and consequently the radicals a_1' and a_1'' remain independent of each other; but for the same reason, the radical a_2'' may be changed to the expression just now given; so that the proposed function of the second order,

$$x = \frac{-a_1}{3} + a_1'' + a_2'',$$

may, by the mere *definitions* of its radicals, and even without attending to the cubic equation which it was designed to satisfy, be put under the form

$$x = \frac{-a_1}{3} + a_1'' + \sqrt[3]{1} \, . \, \frac{c_2}{a_1''},$$

the number of prime roots of variables being depressed from three to two; and consequently that proposed function was not *irreducible* in the sense which has been already explained.

[6.] From the foregoing properties of irrational and irreducible functions, it follows easily that if any *one* value of any such function $b^{(m)}$, corresponding to any one system of values of the radicals on which it depends, be equal to any one root of any equation of the form

$$x^s + \mathrm{A}_1 x^{s-1} + \cdots + \mathrm{A}_{s-1} x + \mathrm{A}_s = 0,$$

in which the coefficients $\mathrm{A}_1, \ldots \mathrm{A}_s$ are any rational functions of the n original quantities $a_1, \ldots a_n$; in such a manner that for some one system of values of the radicals a_1', &c., the equation

$$b^{(m)s} + \mathrm{A}_1 b^{(m)s-1} + \cdots + \mathrm{A}_s = 0$$

is satisfied: then the same equation must be satisfied, also, for *all* systems of values of those radicals, consistent with their equations of definition. It is an immediate consequence of this result, that all the values of the function which has already been denoted by the symbol $b^{(m)}_{\gamma_1^{(m)}, \ldots \gamma^{(m)}_{n^{(m)}}}$ must represent roots of the same equation of the s^{th} degree; and the same principles show that all these values of $b^{(m)}_{\gamma_1^{(m)} \ldots}$ must be *unequal* among themselves, and therefore must represent so many *different roots* $x_1, x_2, \ldots$ of the same equations $x^s +$&c. $= 0$, if every index or exponent $\gamma_i^{(m)}$ be restricted, as before, to denote either zero or some positive integer number less than the corresponding exponent $\alpha_i^{(m)}$: for if, with this restriction, any two of the values of $b^{(m)}_{\gamma_1^{(m)}, \ldots}$ could be supposed equal, an equation of condition between the radicals $a_1^{(m)}$, &c. would arise, which would be inconsistent with the supposed irreducibility of the function $b^{(m)}$.

For example, having found that the cubic equation

$$x^3 + a_1 x^2 + a_2 x + a_3 = 0$$

is satisfied by the irrational and irreducible function b'' above assigned, we can infer that the same equation is satisfied by all the three values b_0'', b_1'', b_2'' of the function $b''_{\gamma_1''}$; and that these three values must be all unequal among themselves, so that they must represent some three unequal roots x_1, x_2, x_3, and consequently all the three roots of the cubic equation proposed.

[7.] Combining the result of the last article with that which was before obtained respecting the isolating of a term of a development, we see that if any root x of any proposed equation, of any degree s, in which the s coefficients $\mathrm{A}_1, \ldots \mathrm{A}_s$ are still supposed to be rational functions of the n original quantities $a_1, \ldots a_n$, can be expressed as an irrational and irreducible function $b^{(m)}$ of those original quantities; and if that function $b^{(m)}$ be developed under the form above assigned; then every term $t^{(m)}_{\beta_1^{(m)}, \ldots}$ of this development may be expressed as a rational (and indeed linear) function of some or all of the s roots $x_1, x_2, \ldots x_s$ of the same proposed equation.

For example, when we have found that a root x of the cubic equation

$$x^3 + a_1 x^2 + a_2 x + a_3 = 0$$

can be represented by the irrational and irreducible function already mentioned,

$$x = b'' = b'_0 + b'_1 a''_1 + b'_2 a''^2_1 = t''_0 + t''_1 + t''_2,$$

(in which $b'_1 = 1$,) we can express the separate terms of this last development as follows,

$$\begin{aligned} t''_0 &= b'_0 = \tfrac{1}{3}(x_1 + x_2 + x_3), \\ t''_1 &= b'_1 a''_1 = \tfrac{1}{3}(x_1 + \rho_3^{-1} x_2 + \rho_3^{-2} x_3), \\ t''_2 &= b'_2 a''^2_1 = \tfrac{1}{3}(x_1 + \rho_3^{-2} x_2 + \rho_3^{-4} x_3); \end{aligned}$$

namely, by changing b''_0, b''_1, b''_2 to x_1, x_2, x_3 in the expressions found before for t''_0, t''_1, t''_2.

In like manner, when a root x of the biquadratic equation

$$x^4 + a_1 x^3 + a_2 x^2 + a_3 x + a_4 = 0$$

is represented by the irrational function

$$\begin{aligned} x = b''' &= b''_{0,0} + b''_{1,0} a'''_1 + b''_{0,1} a'''_2 + b''_{1,1} a'''_1 a'''_2 \\ &= t'''_{0,0} + t'''_{1,0} + t'''_{0,1} + t'''_{1,1}, \end{aligned}$$

in which $b''_{1,0} = b''_{0,1} = 1$, we easily derive, from results obtained before, (by merely changing $b'''_{0,0}$, $b'''_{0,1}$, $b'''_{1,0}$, $b'''_{1,1}$ to x_1, x_2, x_3, x_4,) the following expressions for the four separate terms of this development:

$$\begin{aligned} t'''_{0,0} &= b''_{0,0} = \tfrac{1}{4}(x_1 + x_2 + x_3 + x_4), \\ t'''_{1,0} &= b''_{1,0} a'''_1 = \tfrac{1}{4}(x_1 + x_2 - x_3 - x_4), \\ t'''_{0,1} &= b''_{0,1} a'''_2 = \tfrac{1}{4}(x_1 - x_2 + x_3 - x_4), \\ t'''_{1,1} &= b''_{1,1} a'''_1 a'''_2 = \tfrac{1}{4}(x_1 - x_2 - x_3 + x_4); \end{aligned}$$

x_1, x_2, x_3, x_4 being some four unequal roots, and therefore all the four roots of the proposed biquadratic equation.

And when that equation has a root represented in this other way, which also has been already indicated, and in which $b'''_1 = 1$,

$$x = b^{\mathrm{IV}} = \frac{-a_1}{4} + a'''_1 + a^{\mathrm{IV}}_1 = b'''_0 + b'''_1 a^{\mathrm{IV}}_1 = t^{\mathrm{IV}}_0 + t^{\mathrm{IV}}_1,$$

then each of the two terms of this development may be separately expressed as follows,

$$\begin{aligned} t^{\mathrm{IV}}_0 &= b'''_1 = \tfrac{1}{2}(x_1 + x_2), \\ t^{\mathrm{IV}}_1 &= b'''_1 a^{\mathrm{IV}}_1 = \tfrac{1}{2}(x_1 - x_2), \end{aligned}$$

x_1 and x_2 being some two unequal roots of the same biquadratic equation.

A still more simple example is supplied by the quadratic equation,

$$x^2 + a_1 x + a_2 = 0;$$

for when we represent a root x of this equation as follows,

$$x = b' = \frac{-a_1}{2} + a'_1 = t'_0 + t'_1,$$

we have the following well-known expressions for the two terms t'_0, t'_1, as rational and linear functions of the roots x_1, x_2,

$$\begin{aligned} t'_0 &= \frac{-a_1}{2} = \tfrac{1}{2}(x_1 + x_2), \\ t'_1 &= a'_1 = \tfrac{1}{2}(x_1 - x_2). \end{aligned}$$

In these examples, the radicals of highest order, namely a'_1 in b', a''_1 in b'', a'''_1 and a'''_2 in b''', and a^{IV}_1 in b^{IV}, have all had the coefficients of their first powers equal to unity; and consequently have been themselves expressed as rational (though unsymmetric) functions of the roots of that equation in x, which the function $b^{(m)}$ satisfies; namely,

$$\begin{aligned} a'_1 &= \tfrac{1}{2}(x_1 - x_2), \\ a''_1 &= \tfrac{1}{3}(x_1 + \rho_3^2 x_2 + \rho_3 x_3), \\ a'''_1 &= \tfrac{1}{4}(x_1 + x_2 - x_3 - x_4), \\ a'''_2 &= \tfrac{1}{4}(x_1 - x_2 + x_3 - x_4), \\ a^{\mathrm{IV}}_1 &= \tfrac{1}{2}(x_1 - x_2); \end{aligned}$$

the first expression being connected with the general quadratic, the second with the general cubic, and the three last with the general biquadratic equation. We shall soon see that all these results are included in one more general.

[8.] To illustrate, by a preliminary example, the reasonings to which we are next to proceed, let it be supposed that any two of the terms $t^{(m)}_{\beta^{(m)}_1,\dots}$ are of the forms

$$t''_{2,1,3,4} = b'_{2,1,3,4} a''^2_1 a''_2 a''^3_3 a''^4_4,$$

and

$$t''_{1,1,2,3} = b'_{1,1,2,3} a''_1 a''_2 a''^2_3 a''^3_4,$$

in which the radicals are defined by equations such as the following

$$a''^3_1 = f'_1, \quad a''^3_2 = f'_2, \quad a''^5_3 = f'_3, \quad a''^5_4 = f'_4,$$

their exponents α''_1, α''_2, α''_3, α''_4 being respectively equal to the numbers 3, 3, 5, 5. We shall then have, by raising the two terms t'' to suitable powers, and attending to the equations of definition, the following expressions:

$$\begin{aligned} t''^{10}_{2,1,3,4} &= b'^{10}_{2,1,3,4} f'^6_1 f'^3_2 f'^6_3 f'^8_4 a''^2_1 a''_2; \\ t''^{10}_{1,1,2,3} &= b'^{10}_{1,1,2,3} f'^3_1 f'^3_2 f'^4_3 f'^6_4 a''_1 a''_2; \\ t''^{6}_{2,1,3,4} &= b'^{6}_{2,1,3,4} f'^4_1 f'^2_2 f'^3_3 f'^4_4 a''^3_3 a''^4_4; \\ t''^{6}_{1,1,2,3} &= b'^{6}_{1,1,2,3} f'^2_1 f'^2_2 f'^2_3 f'^3_4 a''^2_3 a''^3_4; \end{aligned}$$

which give

$$\mathrm{T}''_1 = \mathrm{C}'_1 a''_1, \quad \mathrm{T}''_2 = \mathrm{C}'_2 a''_2, \quad \mathrm{T}''_3 = \mathrm{C}'_3 a''_3, \quad \mathrm{T}''_4 = \mathrm{C}'_4 a''_4,$$

if we put, for abridgment,

$$\begin{aligned}
\mathrm{T}''_1 &= t''^{10}_{2,1,3,4} t''^{-10}_{1,1,2,3}; & \mathrm{C}'_1 &= b'^{10}_{2,1,3,4} b'^{-10}_{1,1,2,3} f'^3_1 f'^2_3 f'^2_4;\\
\mathrm{T}''_2 &= t''^{-10}_{2,1,3,4} t''^{20}_{1,1,2,3}; & \mathrm{C}'_2 &= b'^{-10}_{2,1,3,4} b'^{20}_{1,1,2,3} f'^3_2 f'^2_3 f'^4_4;\\
\mathrm{T}''_3 &= t''^{18}_{2,1,3,4} t''^{-24}_{1,1,2,3}; & \mathrm{C}'_3 &= b'^{18}_{2,1,3,4} b'^{-24}_{1,1,2,3} f'^4_1 f'^{-2}_2 f'_3;\\
\mathrm{T}''_4 &= t''^{-12}_{2,1,3,4} t''^{18}_{1,1,2,3}; & \mathrm{C}'_4 &= b'^{-12}_{2,1,3,4} b'^{18}_{1,1,2,3} f'^{-2}_1 f'^2_2 f'_4.
\end{aligned}$$

And, with a little attention, it becomes clear that the same sort of process may be applied to the terms $t^{(m)}_{\beta^{(m)}_1,\dots}$ of the development of any irreducible function $b^{(m)}$; so that we have, in general, a system of relations, such as the following:

$$\mathrm{T}^{(m)}_1 = \mathrm{C}^{(m-1)}_1 a^{(m)}_1; \quad \dots \quad \mathrm{T}^{(m)}_{n^{(m)}} = \mathrm{C}^{(m-1)}_{n^{(m)}} a^{(m)}_{n^{(m)}};$$

in which $\mathrm{T}^{(m)}_i$ is the product of certain powers (with exponents positive, or negative, or null) of the various terms $t^{(m)}_{\beta^{(m)}_1,\dots}$; and the coefficient $\mathrm{C}^{(m-1)}_i$ is different from zero, but is of an order lower than m. For if any radical of the order m were supposed to be so inextricably connected, in every term, with one or more of the remaining radicals of the same highest order, that it could not be disentangled from them by a process of the foregoing kind; and that thus the foregoing *analysis* of the function $b^{(m)}$ should be unable to conduct to separate expressions for those radicals; it would then, reciprocally, have been unnecessary to calculate them separately, in effecting the *synthesis* of that function; which function, consequently, would not be irreducible. If, for example, the exponents $\alpha^{(m)}_1$ and $\alpha^{(m)}_2$, which enter into the equations of definition of the radicals $a^{(m)}_1$ and $a^{(m)}_2$ should both be $= 3$, so that those radicals should both be cube-roots of functions of lower orders; and if these two cube-roots should enter only by their product, so that no analysis of the foregoing kind could obtain them otherwise than in connexion, and under the form $\mathrm{C}^{(m-1)} a^{(m)}_1 a^{(m)}_2$; it would then have been sufficient, in effecting the synthesis of $b^{(m)}$, to have calculated only the cube-root of the product $a^{(m)3}_1 a^{(m)3}_2 = f^{(m-1)}_1 f^{(m-1)}_2 = f^{\backslash(m-1)}$, instead of calculating separately the cube-roots of its two factors $a^{(m)3}_1 = f^{(m-1)}$, and $a^{(m)3}_2 = f^{(m-1)}_2$: the number of extractions of prime roots of variables might, therefore, have been diminished in the calculation of the function $b^{(m)}$, which would be inconsistent with the irreducibility of that function.

In the cases of the irreducible functions b', b'', b''', b^{IV}, which have been above assigned, as representing roots of the general quadratic, cubic, and biquadratic equations, the theorem of the present article is seen at once to hold good; because in these the radicals of highest order are themselves terms of the developments in question, the coefficients of their first powers being already equal to unity. Thus in the development of b', we have $a'_1 = t'_1$; in b'', we have $a''_1 = t''_1$; in b''', we have $a'''_1 = t'''_{1,0}$, and $a'''_2 = t'''_{0,1}$; and in b^{IV}, we have $a^{\mathrm{IV}}_1 = t^{\mathrm{IV}}_1$.

[9.] By raising to the proper powers the general expressions of the form

$$\mathrm{T}^{(m)}_i = \mathrm{C}^{(m-1)}_i a^{(m)}_i,$$

we obtain a system of $n^{(m)}$ equations of this other form

$$\mathrm{T}_i^{(m)\alpha_i^{(m)}} = \mathrm{C}_i^{(m-1)\alpha_i^{(m)}} f_i^{(m-1)} = f_i'^{(m-1)},$$

$f_i'^{(m-1)}$ being some new irrational function, of an order lower than m; and by combining the same expressions with those which define the various terms $t^{(m)}_{\beta_1^{(m)},\ldots}$, the number of which terms we shall denote by the symbol $t^{(m)}$, we obtain another system of $t^{(m)}$ equations, of which the following is a type,

$$\mathrm{U}^{(m-1)}_{\beta_1^{(m)},\ldots\,\beta^{(m)}_{n^{(m)}}} = b'^{(m-1)}_{\beta_1^{(m)},\ldots\,\beta^{(m)}_{n^{(m)}}},$$

if we put, for abridgment,

$$\mathrm{U}^{(m-1)}_{\beta_1^{(m)},\ldots} = t^{(m)}_{\beta_1^{(m)},\ldots} \,.\, \mathrm{T}_1^{(m)-\beta_1^{(m)}} \cdots \mathrm{T}_{n^{(m)}}^{(m)-\beta^{(m)}_{n^{(m)}}},$$

and

$$b'^{(m-1)}_{\beta_1^{(m)},\ldots} = b^{(m-1)}_{\beta_1^{(m)},\ldots} \,.\, \mathrm{C}_1^{(m-1)-\beta_1^{(m)}} \cdots \mathrm{C}_{n^{(m)}}^{(m-1)-\beta^{(m)}_{n^{(m)}}}.$$

In this manner we obtain in general $n^{(m)} + t^{(m)}$ equations, in each of which the product of certain powers, (with positive, negative, or null exponents,) of the $t^{(m)}$ terms of the development of the irrational function $b^{(m)}$, is equated to some other irrational function, $f'^{(m-1)}$ or $b'^{(m-1)}$, of an order lower than m. Indeed, it is to be observed, that since these various equations are obtained by an elimination of the $n^{(m)}$ radicals of highest order, between their $n^{(m)}$ equations of definition and the $t^{(m)}$ expressions for the $t^{(m)}$ terms of the development of $b^{(m)}$, they cannot be equivalent to more than $t^{(m)}$ distinct relations. But, among them, they must involve explicitly all the radicals of lower orders, which enter into the composition of the irreducible function $b^{(m)}$. For if any radical $a_i^{(k)}$, of order lower than m, were wanting in all the $n^{(m)} + t^{(m)}$ functions of the forms

$$f_i'^{(m-1)} \quad \text{and} \quad b'^{(m-1)}_{\beta_1^{(m)},\ldots},$$

we might then employ instead of the old system of radicals $a_1^{(m)}, \ldots$ of the order m, a new and equally numerous system of radicals $a_1'^{(m)}, \ldots$ according to the following type,

$$a_i'^{(m)} = \mathrm{T}_i^{(m)} = \sqrt[\alpha_i^{(m)}]{f_i'^{(m-1)}};$$

and might then express all the $t^{(m)}$ terms of $b^{(m)}$, by means of these new radicals, according to the formula

$$t^{(m)}_{\beta_1^{(m)},\ldots} = b'^{(m-1)}_{\beta_1^{(m)},\ldots} \,.\, a_1'^{(m)\beta_1^{(m)}} \ldots a_{n^{(m)}}'^{(m)\beta^{(m)}_{n^{(m)}}},$$

which would not involve the radical $a_i^{(k)}$; so that in this way the number of extractions of prime roots of variables might be diminished, which would be inconsistent with the irreducibility of $b^{(m)}$.

The results of the present article may be exemplified in the case of any one of the functions b', b'', b''', b^{IV}, which have already been considered. Thus, in the case of the function b'', which represents a root of the general cubic equation, we have

$$\mathrm{T}_1'' = t_1'', \quad \mathrm{C}_1'' = 1, \quad f_1^{\backslash\prime} = f_1', \quad \beta_{\beta_1''}^{\backslash\prime} = b_{\beta_1''}', \quad \mathrm{U}_{\beta_1''}' = t_{\beta_1''}'' \,.\, t_1^{\prime\prime-\beta_1''},$$

and the $n^{(m)} + t^{(m)} = 1 + 3 = 4$ following relations hold good:

$$t_1^{\prime\prime 3} = f_1', \quad t_0'' = b_0', \quad 1 = b_1', \quad t_2'' t_1^{\prime\prime -2} = b_2';$$

of which indeed the third is identically true, and the second does not involve a_1', because $b_0' = -\dfrac{a_1}{3}$; but both the first and fourth of these relations involve that radical a_1', because $f_1' = c_1 + a_1'$, and $b_2' = \dfrac{c_1 - a_1'}{c_2^2}$.

[10.] Since each of the $t^{(m)}$ terms of the development of $b^{(n)}$ can be expressed as a rational function of the s roots $x_1, \ldots x_s$ of that equation of the s^{th} degree which $b^{(m)}$ is supposed to satisfy; it follows that every rational function of these $t^{(m)}$ terms must be likewise a rational function of those s roots, and must admit, as such, of some finite number r of values, corresponding to all possible changes of arrangement of the same s roots amongst themselves. The same term or function must, for the same reason, be itself a root of an equation of the r^{th} degree, of which the coefficients are symmetrical functions of the s roots, $x_1, \ldots x_s$, and therefore are rational functions of the s coefficients $\mathrm{A}_1, \ldots \mathrm{A}_s$, and ultimately of the n original quantities $a_1, \ldots a_n$; while the $r - 1$ other roots of this new equation are the $r - 1$ other values of the same function of $x_1, \ldots x_s$, corresponding to the changes of arrangement just now mentioned. Hence, every one of the $n^{(m)} + t^{(m)}$ functions $\mathrm{T}_i^{(m)\alpha_i^{(m)}}$ and $\mathrm{U}_{\beta_1^{(m)},\ldots}^{(m-1)}$, and therefore also every one of the $n^{(m)} + t^{(m)}$ functions $f_i^{\backslash(m-1)}$ and $b_{\beta_1^{(m)},\ldots}^{\backslash(m-1)}$, to which they are respectively equal, and which have been shown to contain, among them, all the radicals of orders lower than m, must be a root of some such new equation, although the degree r will not in general be the same for all. Treating these new equations and functions, and the radicals of the order $m - 1$, as the equation $x^s + \&\text{c.} = 0$, the function $b^{(m)}$, and the radicals of the order m have already been treated; we obtain a new system of relations, analogous to those already found, and capable of being thus denoted:

$$\begin{aligned} \mathrm{T}_i^{(m-1)} &= \mathrm{C}_i^{(m-2)} a_i^{(m-1)}; \\ \mathrm{T}_i^{(m-1)\alpha_i^{(m-1)}} &= f_i^{\backslash(m-2)}; \\ \mathrm{U}_{\beta_1^{(m-1)},\ldots}^{(m-1)} &= b_{\beta_1^{(m-1)},\ldots}^{\backslash(m-2)}. \end{aligned}$$

And so proceeding, we come at last to a system of the form,

$$\mathrm{T}_1' = \mathrm{C}_1 a_1', \quad \ldots \quad \mathrm{T}_{n'}' = \mathrm{C}_{n'} a_{n'}';$$

in which the coefficient C_i is different from zero, and is a rational function of the n original quantities $a_1, \dots a_n$; while T'_i is a rational function of the s roots $x_1, \dots x_s$ of that equation of the s^{th} degree in x which it has been suppose that $b^{(m)}$ satisfies. We have therefore the expression

$$a'_i = \frac{\mathrm{T}'_1}{\mathrm{C}_i};$$

which enables us to consider every radical a'_i of the first order, as a rational function F'_i of the s roots $x_1, \dots x_s$, and of the n original quantities $a_1, \dots a_n$: so that we may write

$$a'_i = \mathrm{F}'_i(x_1, \dots x_s, a_1, \dots a_n).$$

But before arriving at the last mentioned system of relations, another system of the form

$$\mathrm{T}''_1 = \mathrm{C}'_1 a''_1, \quad \dots \quad \mathrm{T}''_{n''} = \mathrm{C}'_{n''} a''_{n''}$$

must have been found, in which the coefficient C'_i is different from zero, and is a rational function of $a'_1, \dots a'_{n'}$ and of $a_1, \dots, a_n$, while T''_i is a rational function of $x_1, \dots x_s$; we have therefore the expression

$$a''_i = \frac{\mathrm{T}''_i}{\mathrm{C}'_i},$$

and we see that every radical of the second order also is equal to a rational function of $x_1, \dots x_s$ and of $a_1, \dots a_n$: so that we may write

$$a''_i = \mathrm{F}''_i(x_1, \dots x_s, a_1, \dots a_n).$$

And re-ascending thus, through orders higher and higher, we find, finally, by similar reasonings, that every one of the $n' + n'' + \cdots + +n^{(k)} + \cdots + n^{(m)}$ radicals which enter into the composition of the irrational and irreducible function $b^{(m)}$, such as the radical $a_i^{(k)}$, must be expressible as a rational function $\mathrm{F}_i^{(k)}$ of the roots $x_1, \dots x_s$, and of the original quantities $a_1, \dots a_n$: so that we have a complete system of expressions, for all these radicals, which are included in the general formula

$$a_i^{(k)} = \mathrm{F}_i^{(k)}(x_1, \dots x_s, a_1, \dots a_n).$$

Thus, in the case of the cubic equation and the function b'', when we have arrived at the relation

$$t''^3_1 = f'_1,$$

in which

$$t''_1 = \tfrac{1}{3}(x_1 + \rho_3^2 x_2 + \rho_3 x_3), \quad \text{and} \quad f'_1 = c_1 + a'_1,$$

we find that the rational function

$$t''^3_1 = \tfrac{1}{27}(x_1 + \rho_3^2 x_2 + \rho_3 x_3)^3$$

admits only of *two* different values, in whatever way the arrangement of the three roots x_1, x_2, x_3 may be changed; it must therefore be itself a root of a quadratic equation, in which the coefficients are symmetric functions of those three roots, and consequently rational functions of a_1, a_2, a_3; namely, the equation

$$\begin{aligned}0 &= (t_1''^3)^2 - \tfrac{1}{27}\{(x_1+\rho_3^2x_2+\rho_3x_3)^3 + (x_1+\rho_3^2x_3+\rho_3x_2)^3\}(t_1''^3)\\ &\quad + \tfrac{1}{729}(x_1+\rho_3^2x_2+\rho_3x_3)^3(x_1+\rho_3^2x_3+\rho_3x_2)^3\\ &= (t_1''^3)^2 + \tfrac{1}{27}(2a_1^3 - 9a_1a_2 + 27a_3)(t_1''^3) + \left(\frac{a_1^2-3a_2}{9}\right)^3.\end{aligned}$$

The same quadratic equation must therefore be satisfied when we substitute for $t_1''^3$ the function $x_1 + a_1'$ to which it is equal, and in which a_1' is a square root; it must therefore be satisfied by *both* values of the function $c_1 \pm a_1'$, because the radical a_1' must be subject to no condition except that by which its square is determined; therefore, this radical a_1' must be equal to the semi-difference of two unequal roots of the same quadratic equation; that is, to the semi-difference of the two values of the rational function $t_1''^3$; which semi-difference *is itself a rational function of* x_1, x_2, x_3, namely

$$\begin{aligned}a_1' &= \tfrac{1}{54}\{(x_1+\rho_3^2x_2+\rho_3x_3)^3 - (x_1+\rho_3^2x_3+\rho_3x_2)^3\}\\ &= \tfrac{1}{18}(\rho_3^2-\rho_3)(x_1-x_2)(x_1-x_3)(x_2-x_3) = \mathrm{F}_1'(x_1,x_2,x_3).\end{aligned}$$

The same conclusion would have been obtained, though in a somewhat less simple way, if we had employed the relation

$$t_2''t_1''^{-2} = b_2',$$

in which

$$t_2''t_1''^{-2} = \frac{3(x_1+\rho_3^2x_3+\rho_3x_2)}{(x_1+\rho_3^2x_2+\rho_3x_3)^2}, \quad b_2' = \frac{c_1 - a_1'}{c_2^2}.$$

[11.] In general, let p be the number of values which the rational function $\mathrm{F}_i^{(k)}$ can receive, by altering in all possible ways the arrangements of the s roots $x_1, \ldots x_s$, these roots being still treated as arbitrary and independent quantities, (so that p is equal either to the product $1\,.\,2\,.\,3 \ldots s$, or to some submultiple of that product); we shall then have an *identical* equation of the form

$$\mathrm{F}_i^{(k)p} + \mathrm{D}_1\mathrm{F}_i^{(k)p-1} + \cdots + \mathrm{D}_{p-1}\mathrm{F}_i^{(k)} + \mathrm{D}_p = 0,$$

in which the coefficients $\mathrm{D}_1, \ldots \mathrm{D}_p$ are rational functions of $a_1, \ldots a_n$; and therefore at least *one* value of the radical $a_i^{(k)}$ must satisfy the equation

$$a_i^{(k)p} + \mathrm{D}_1 a_i^{(k)p-1} + \cdots + \mathrm{D}_{p-1}a_i^{(k)} + \mathrm{D}_p = 0.$$

But in order to do this, it is necessary, for reasons already explained, that *all* values of the same radical $a_i^{(k)}$, obtained by multiplying itself and all its subordinate radicals of the same functional system by any powers of the corresponding roots of unity, should satisfy the same equation; and therefore that the number q of these values of the radical $a_i^{(k)}$ should *not exceed* the degree p of that equation, or the number of the values of the rational function $\mathrm{F}_i^{(k)}$.

Again, since we have denoted by q the number of values of the radical, we must suppose that it satisfies identically an equation of the form

$$a_i^{(k)q} + \mathrm{E}_1 a^{(k)q-1} + \cdots + \mathrm{E}_{q-1} a_i^{(k)} + \mathrm{E}_q = 0,$$

the coefficients $\mathrm{E}_1, \ldots \mathrm{E}_q$ being rational functions of $a_1, \ldots a_n$; and therefore that at least one value of the function $\mathrm{F}_i^{(k)}$ satisfies the equation

$$\mathrm{F}_i^{(k)q} + \mathrm{E}_1 \,.\, \mathrm{F}_i^{(k)q-1} + \cdots + \mathrm{E}_{q-1}\mathrm{F}_i^{(k)} = 0.$$

Suppose now that the s roots $x_1, \ldots x_s$ of the original equation in x,

$$x^s + \mathrm{A}_1 x^{s-1} + \cdots + \mathrm{A}_{s-1} x + \mathrm{A}_s = 0,$$

are really unconnected by any relation among themselves, a supposition which requires that s should not be greater than n, since $\mathrm{A}_1, \ldots \mathrm{A}_s$ are rational functions of $a_1, \ldots a_n$; suppose also that $a_1, \ldots a_n$ can be expressed, reciprocally, as rational functions of $\mathrm{A}_1, \ldots \mathrm{A}_s$, a supposition which requires, reciprocally, that n should not be greater than s, because the *original quantities* $a_1, \ldots a_n$ *are, in this whole discussion, considered as independent of each other.* With these suppositions, which involve the equality $s = n$, we may consider the n quantities $a_1, \ldots a_n$, and therefore also the q coefficients $\mathrm{E}_1, \ldots \mathrm{E}_q$, as being symmetric functions of the n roots $x_1, \ldots x_n$ of the equation

$$x^n + \mathrm{A}_1 x^{n-1} + \cdots + \mathrm{A}_{n-1} x + \mathrm{A}_n = 0;$$

we may also consider $\mathrm{F}_i^{(k)}$ as being a rational but unsymmetric function of the same n arbitrary roots, so that we may write

$$a_i^k = \mathrm{F}_i^{(k)}(x_1, \ldots x_n);$$

and since the truth of the equation

$$\mathrm{F}^{(k)q} + \mathrm{E}_1 \mathrm{F}_i^{(k)q-1} + \cdots + \mathrm{E}_q = 0$$

must depend only on the *forms of the functions*, and not on the *values of the quantities* which it involves, (those values being altogether arbitrary,) we may alter in any manner the arrangement of those n arbitrary quantities $x_1, \ldots x_n$, and the equation must still hold good. But by such changes of arrangement, the symmetric coefficients $\mathrm{E}_1, \ldots \mathrm{E}_q$ remain unchanged, while the rational but unsymmetric function $\mathrm{F}_i^{(k)}$ takes, in succession, all those p values of which it was before supposed to be capable; thse p unequal values therefore must all be roots of the same equation of the q^{th} degree, and consequently q must *not be less* than p. And since it has been shown that the former of these two last mentioned numbers must *not exceed* the latter, it follows that they must be *equal* to each other, so that we have the relation

$$q = p:$$

that is, the radical $a_i^{(k)}$ and the rational function $\text{F}_i^{(k)}$ must be exactly *coextensive in multiplicity of value.*

For example, when, in considering the irreducible irrational expression b'' for a root of the general cubic, we are conducted to the relation assigned in the last article,

$$a'_1 = \text{F}'_1(x_1, x_2, x_3) = \tfrac{1}{18}(\rho_3^3 - \rho_3)(x_1 - x_2)(x_1 - x_3)(x_2 - x_3);$$

we can then at pleasure infer, either that the radical a'_i must admit (as a radical) of two and only two values, if we have previously perceived that the rational function F'_1 admits (as a rational function) of two values, and only two, corresponding to changes of arrangement of the three roots x_1, x_2, x_3, namely, the two following values, which differ by their signs,

$$\pm\tfrac{1}{18}(\rho_3^3 - \rho_3)(x_1 - x_2)(x_1 - x_3)(x_2 - x_3);$$

or else we may infer that the function F'_1 admits thus of two values and two only, for all changes of arrangement of x_1, x_2, x_3, if we have perceived that the radical a'_1 (as being given by its square,

$$a_1'^2 = f_1 = c_1^2 - c_2^3,$$

which square is rational,) admits, itself, of the two values $\pm a'_1$ which differ in their signs.

[12.] The conditions assumed in the last article are all fulfilled, when we suppose the coefficients A_1 &c. to coincide with the n original quantities a_1, &c., that is, when we return to the equation originally proposed;

$$x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n = 0,$$

which is the general equation of the n^{th} degree: so that we have, for any radical $a_i^{(k)}$, which enters into the composition of any irrational and irreducible function representing any root of any such equation, an expression of the form

$$a_i^{(k)} = \text{F}_i^{(k)}(x_1, \ldots x_n);$$

the radical and the rational function being coextensive in multiplicity of value. We are, therefore, conducted thus to the following important theorem, to which Abel first was led, by reasonings somewhat different from the foregoing: namely, that "if a root x of the general equation of any particular degree n can be expressed as an irreducible irrational function $b^{(m)}$ of the n arbitrary coefficients of that equation, then every radical $a_i^{(k)}$, which enters into the composition of that function $b^{(m)}$, must admit of being expressed as a rational, though unsymmetric function $\text{F}_i^{(k)}$ of the n arbitrary roots of the same general equation; and this rational but unsymmetric function $\text{F}_i^{(k)}$ must admit of receiving exactly the same variety of values, through changes of arrangement of the n roots on which it depends, as that which the radical $a_i^{(k)}$ can receive, through multiplications of itself and of all its subordinate functional radicals by any powers of the corresponding roots of unity."

Examples of the truth of this theorem have already been given, by anticipation, in the seventh and tenth articles of this Essay; to which we may add, that the radicals a''_i and a'_i, in the expressions given above for a root of the general biquadratic, admit of being thus expressed:

$$
\begin{aligned}
a_1'' &= \tfrac{1}{48}\{(x_1+x_2-x_3-x_4)^2 + \rho_3^2(x_1-x_2+x_3-x_4)^2 + \rho_3(x_1-x_2-x_3+x_4)^2\} \\
&= \tfrac{1}{12}\{x_1x_2+x_3x_4+\rho_3^2(x_1x_3+x_2x_4)+\rho_3(x_1x_4+x_2x_3)\}; \\
a_1' &= \tfrac{1}{3456}\{x_1x_2+x_3x_4+\rho_3^2(x_1x_3+x_2x_4)+\rho_3(x_1x_4+x_2x_3)\}^3 \\
&\quad -\tfrac{1}{3456}\{x_1x_2+x_3x_4+\rho_3^2(x_1x_4+x_2x_3)+\rho_3(x_1x_3+x_2x_4)\}^3 \\
&= \tfrac{1}{1152}(\rho_3^2-\rho_3)(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_2-x_3)(x_2-x_4)(x_3-x_4).
\end{aligned}
$$

But before we proceed to apply this theorem to prove, in a manner similar to that of ABEL, the impossibility of obtaining any finite expression, irrational and irreducible, for a root of the general equation of the fifth degree, it will be instructive to apply it, in a new way, (according to the announcement made in the second article,) to equations of lower degrees; so as to draw, from those lower equations, a class of illustrations quite different from those which have been heretofore adduced: namely, by showing, *à priori*, with the help of the same general theorem, that no new finite function, irrational and irreducible, can be found, essentially distinct in its radicals from those which have long since been discovered, for expressing any root of any such lower but general equation, quadratic, cubic or biquadratic, in terms of the coefficients of that equation.

[13.] Beginning then with the general quadratic,

$$x^2 + a_1x + a_2 = 0,$$

let us endeavour to investigate, *à priori*, with the help of the foregoing theorem, all possible forms of irrational and irreducible functions $b^{(m)}$, which can express a root x of this quadratic, in terms of the two arbitrary coefficients a_1, a_2, so as to satisfy identically, or independently of the values of those two coefficients, the equation

$$b^{(m)2} + a_1b^{(m)} + a_2 = 0.$$

The two roots of the proposed quadratic being denoted by the symbols x_1 and x_2, we know that the two coefficients a_1 and a_2 are equal to the following symmetric functions,

$$a_1 = -(x_1+x_2), \quad a_2 = x_1x_2;$$

we cannot therefore suppose either root to be a rational function b of these coefficients, because an unsymmetric function of two arbitrary quantities cannot be equal to a symmetric function of the same; and consequently we must suppose that the exponent m of the order of the sought function $b^{(m)}$ is greater than 0. The expression $b^{(m)}$ for x must therefore involve at least one radical a_1', which must itself admit of being expressed as a rational but unsymmetric function of the two roots x_1, x_2,

$$a_1' = \mathrm{F}_1'(x_1, x_2),$$

and of which some prime power can be expressed as a rational function of the two coefficients a_1, a_2,

$$a_1'^{\alpha_1'} = f(a_1, a_2),$$

the exponent α_1' being equal to the number of the values

$$\mathrm{F}_1'(x_1, x_2), \quad \mathrm{F}_1'(x_2, x_1),$$

of the unsymmetric function F_1', and consequently being $= 2$; so that the radical a_1' must be a square root, and must have two values differing in sign, which may be thus expressed:

$$+a_1' = \mathrm{F}_1'(x_1, x_2), \quad -a_1' = \mathrm{F}_1'(x_2, x_1).$$

But, in general, whatever rational function may be denoted by F, the quotients

$$\frac{\mathrm{F}(x_1, x_2) + \mathrm{F}(x_2, x_1)}{2} \quad \text{and} \quad \frac{\mathrm{F}(x_1, x_2) - \mathrm{F}(x_2, x_1)}{2(x_1 - x_2)}$$

are some symmetric functions, a and b; so that we may put generally

$$\mathrm{F}(x_1, x_2) = a + b(x_1 - x_2), \quad \mathrm{F}(x_2, x_1) = a - b(x_1 - x_2);$$

therefore, since we have, at present,

$$\mathrm{F}_1'(x_2, x_1) = -\mathrm{F}_1'(x_1, x_2),$$

the function F_1' must be of the form

$$\mathrm{F}_1'(x_1, x_2) = b(x_1 - x_2),$$

the multiplier b being symmetric. At the same time,

$$a_1' = b(x_1 - x_2),$$

and therefore the function f_1 is of the form

$$f_1(a_1, a_2) = a_1'^2 = b^2(x_1 - x_2)^2 = b^2(a_1^2 - 4a_2),$$

so that the radical a_1' may be thus expressed,

$$a_1' = \sqrt{b^2(a_1^2 - 4a_2)},$$

in which, b is some rational function of the coefficients a_1, a_2. No other radical a_2' of the first order can enter into the sought irreducible expression for x; because the same reasoning would show that any such new radical ought to be reducible to the form

$$a_2' = c(x_1 - x_2) = \frac{c}{b} a_1',$$

c being some new symmetric function of the roots, and consequently some new rational function of the coefficients; so that, after calculating the radical a'_1, it would be unnecessary to effect any new extraction of prime roots for the purpose of calculating a'_2, which latter radical would therefore be superfluous. Nor can any radical a''_1 of higher order enter, because such radical would have $2\alpha''_1$ values, α''_1 being greater than 1, while any rational function F''_1, of two arbitrary quantities x_1, x_2, can receive only two values, through any changes of their arrangement. The exponent m, of the order of the sought irreducible function $b^{(m)}$, must therefore be $= 1$, and this function itself must be of the form

$$b' = b_0 + b_1 a'_1,$$

b_0 and b_1 being rational functions of a_1, a_2, or symmetric functions of the two roots x_1, x_2, which roots must admit of being separately expressed as follows:

$$x_1 = b_0 + b_1 a'_1, \quad x_2 = b_0 - b_1 a'_1,$$

if any expression of the sought kind can be found for either of them. It is, therefore, necessary and sufficient for the existence of such an expression, that the two following quantities,

$$b_0 = \frac{x_1 + x_2}{2}, \quad b_1 = \frac{x_1 - x_2}{2a'_1},$$

should admit of being expressed as rational functions of a_1, a_2; and this condition is satisfied, since the foregoing relations give

$$b_0 = -\frac{a_1}{2}, \quad b_1 = \frac{1}{2b}.$$

We find, therefore, as the sought irrational and irreducible expression, and as the only possible expression of that kind, (or at least as one with which all others must essentially coincide,) for a root x of the general quadratic, the following:

$$x = b' = \frac{-a_1}{2} + \frac{1}{2b}\sqrt{b^2(a_1^2 - 4a_2)};$$

b still denoting any arbitrary rational function of the two arbitrary coefficients a_1, a_2, or any numerical constant, (such as the number $\frac{1}{2}$, which was the value of the quantity b in the formulæ of the preceding articles,) and the two separate roots x_1, x_2, being obtained by taking separately the two signs of the radical. And thus we see *à priori*, that *every* method, for calculating a root x of the general quadratic equation as a function of the two coefficients, by any finite number of additions, subtractions, multiplications, divisions, elevations to powers, and extractions of prime radicals, (these last extractions being supposed to be reduced to the smallest possible number,) *must* involve the extraction of some one square-root of the form

$$a'_1 = \sqrt{b^2(a_1^2 - 4a_2)},$$

and must *not* involve the extraction of any other radical. But this square-root a'_1 is not essentially distinct from that which is usually assigned for the solution of the general quadratic: it is therefore impossible to discover any *new* irrational expression, finite and irreducible, for a root of that general quadratic, essentially distinct from the expressions which have long been known: and the only possible difference between the extractions of radicals which are required in any two methods of solution, if neither method require any superfluous extraction, is that these methods may introduce different square factors into the expressions of that quantity or function f_1, of which, in each, the square root a'_1 is to be calculated.

[14.] Proceeding to the general cubic,

$$x^3 + a_1x^2 + a_2x + a_3 = 0,$$

we know, first, that the three coefficients are symmetric functions of the three roots,

$$a_1 = -(x_1 + x_2 + x_3), \quad a_2 = x_1x_2 + x_1x_3 + x_2x_3, \quad a_3 = -x_1x_2x_3,$$

so that we cannot express any one of these three arbitrary roots x_1, x_2, x_3, as a rational function b of the three coefficients a_1, a_2, a_3; we must therefore inquire whether it can be expressed as an irrational function $b^{(m)}$, involving at least one radical a'_1 of the first order, which is to satisfy the two conditions,

$$a_1'^{\alpha'_1} = f_1(a_1, a_2, a_3),$$

and

$$a'_1 = \mathrm{F}'_1(x_1, x_2, x_3);$$

the functions f_1 and F'_1 being rational, and the prime exponent α'_1 being either 2 or 3, because it is to be equal to the number of values of the rational function F'_1 obtained by changing in all possible ways the arrangement of the three roots x_1, x_2, x_3, and therefore must be a divisor of the product $1 \,.\, 2 \,.\, 3 = 6$.

Now by the properties of rational functions of three variables, (of which an investigation shall soon be given, but which it is convenient merely to enunciate here, that the course of the main argument may not be too much interrupted,) no three-valued function of three arbitrary quantities x_1, x_2, x_3, can have a symmetric cube; and the only two-valued functions, which have symmetric squares, are of the form

$$b(x_1 - x_2)(x_1 - x_3)(x_2 - x_3),$$

b being a symmetric but otherwise arbitrary multiplier. We must therefore suppose, that the radical a'_1 is a square-root, and that it may be thus represented:

$$\begin{aligned} a'_1 &= \mathrm{F}'_1(x_1, x_2, x_3) = b(x_1 - x_2)(x_1 - x_3)(x_2 - x_3) \\ &= \surd\{b^2(x_1 - x_2)^2(x_1 - x_3)^2(x_2 - x_3)^2\} \\ &= \surd\{b^2(a_1^2a_2^2 - 4a_1^3a_3 - 4a_2^3 + 18a_1a_2a_3 - 27a_3^2)\} \\ &= \sqrt{-108b^2(c_1^2 - c_2^3)}, \end{aligned}$$

b being here rational with respect to a_1, a_2, a_3, as also are c_1 and c_2, which last have the same meanings here as in the second article; so that the function f_1 is of the form,

$$f_1(a_1, a_2, a_3) = -108b^2(c_1^2 - c_2^3).$$

No other radical of the first order, a'_2, can enter into the sought irreducible expression $b^{(m)}$; because the same reasoning would give

$$a'_2 = c(x_1 - x_2)(x_1 - x_3)(x_2 - x_3) = \frac{c}{b}a'_1,$$

c being rational with respect to a_1, a_2, a_3, so that the radical a'_2 would be superfluous. On the other hand, no expression of the form $b_0 + b_1 a'_1$ can represent the three-valued function x; we must therefore suppose that if the sought expression $b^{(m)}$ exist at all, it is, at lowest, of the second order, and involves at least one radical a''_1, such that

$$a_1''^{\alpha_1''} = (\, f_1' =)\, b_0 + b_1 a_1',$$

and

$$a_1'' = \mathrm{F}_1''(x_1, x_2, x_3);$$

the rational function F''_1 admitting of $2\alpha''_1$ values, and consequently the exponent α''_1 being $= 3$, (since it cannot be $= 2$, because no function of three variables has exactly four values,) so that we must suppose the radical a''_1 to be a cube-root, of the form

$$a_1'' = \sqrt[3]{b_0 + b_1 a_1'},$$

b_0 and b_1 being rational with respect to a_1, a_2, a_3. But in order that a six-valued rational function F''_1, of three arbitrary quantities x_1, x_2, x_3, should have a two-valued cube, it must be of the form

$$\mathrm{F}_1''(x_1, x_2, x_3) = (p_0 + p_1 a_1')(x_1 + \rho_3^2 x_2 + \rho_3 x_3);$$

in which p_0 and p_1 are symmetric, a'_1 has the form recently assigned, and ρ_3 is a root of the numerical equation

$$\rho_3^2 + \rho_3 + 1 = 0;$$

we must therefore suppose that

$$a_1'' = (p_0 + p_1 a_1')(x_1 + \rho_3^2 x_2 + \rho_3 x_3),$$

and

$$b_0 + b_1 a_1' = 27(p_0 + p_1 a_1')^3 \left\{ c_1 + \tfrac{1}{18}(\rho_3^2 - \rho_3)\frac{a_1'}{b} \right\},$$

c_1 retaining here its recent meaning; so that the radical a''_1 may be considered as the cube-root of this last expression. If any other radical a''_2 of the second order could enter into the composition of $b^{(m)}$, it ought, for the same reasons, to be either of the form

$$a_2'' = (q_0 + q_1 a_1')(x_1 + \rho_3^2 x_2 + \rho_3 x_3),$$

or else of the form

$$a_2'' = (q_0 + q_1 a_1')(x_1 + \rho_3 x_2 + \rho_3^2 x_3),$$

ρ_3 being here the same root of the numerical equation $\rho_3^2 + \rho_3 + 1 = 0$, as in the expression for α''_1; we should therefore have either the relation

$$a_2'' = \frac{q_0 + q_1 a_1'}{p_0 + p_1 a_1'} a_1'',$$

or else the relation

$$a_2'' = \frac{9c_2(p_0 + p_1 a_1)(q_0 + q_1 a_1')}{a_1''},$$

c_2 retaining its recent meaning; so that in each case it would be superfluous to perform any new extraction of a cube-root or other radical in order to calculate a_2'', after a_1' and a_1'' had been calculated; and consequently no such other radical a_2'' of the second order can enter into the composition of the irreducible function $b^{(m)}$. If then that function be itself of the second order, it must be capable of being put under the form

$$b'' = b_0' + b_1' a_1'' + b_2' a_1''^2,$$

b_0', b_1', b_2' being functions of the forms

$$b_0' = (b_0')_0 + (b_0')_1 a_1', \quad b_1' = (b_1')_0 + (b_1')_1 a_1', \quad b_2' = (b_2')_0 + (b_2')_1 a_1',$$

in which the radicals a_1' and a_1'' have the forms lately found, and $(b_0')_0, \ldots (b_2')_1$ are rational functions of a_1, a_2, a_3. And on the same supposition, the three roots x_1, x_2, x_3, of that equation must, in some arrangement or other, be represented by the three expressions,

$$\begin{aligned}
x_\alpha &= b_0'' = b_0' + b_1' a_1'' + b_2' a_1''^2, \\
x_\beta &= b_1'' = b_0' + \rho_3 b_1' a_1'' + \rho_3^2 b_2' a_1''^2, \\
x_\gamma &= b_2'' = b_0' + \rho_3^2 b_1' a_1'' + \rho_3 b_2' a_1''^2,
\end{aligned}$$

ρ_3 retaining here its recent value: which expressions reciprocally will be true, if the following relations,

$$\begin{aligned}
b_0' &= \tfrac{1}{3}(x_\alpha + x_\beta + x_\gamma), \\
b_1' a_1'' &= \tfrac{1}{3}(x_\alpha + \rho_3^2 x_\beta + \rho_3 x_\gamma), \\
b_2' a_1''^2 &= \tfrac{1}{3}(x_\alpha + \rho_3 x_\beta + \rho_3^2 x_\gamma),
\end{aligned}$$

can be made to hold good, by any suitable arrangement of the roots x_α, x_β, x_γ, and by any suitable selection of those rational functions of a_1, a_2, a_3, which have hitherto been left undetermined. Now, for this purpose it is necessary and sufficient that the arrangement of the roots x_α, x_β, x_γ, should coincide with one or other of the three following arrangements, namely x_1, x_2, x_3, or x_2, x_3, x_1, or x_3, x_1, x_2; the value of $3b_1'(p_0 + p_1 a_1')$ being, in the first case, unity, in the second case, ρ_3; and, in the third case, ρ_3^2; while, in every case, the value of b_0' is to be $\dfrac{-a_1}{3}$, and that of $b_1' b_2'(b_0 + b_1 a_1')$ is to be c_2. All these suppositions are compatible with the conditions assigned before; nor is there any essential difference between the three cases of arrangement just now mentioned, since the passage from any one to any other may be made (as we have seen) by merely multiplying the coefficient b_1', which admits of an arbitrary multiplier, by an imaginary cube-root of unity. We have, therefore, the following irrational and irreducible expression for the root x of the general cubic, as a function of the second order,

$$x = b'' = \frac{-a_1}{3} + \frac{a_1''}{3(p_0 + p_1 a_1')} + \frac{3c_2(p_0 + p_1 a_1')}{a_1''};$$

in which it is to be remembered that

$$a_1''^3 = 27(p_0 + p_1 a_1')^3 \left\{ c_1 + \tfrac{1}{18}(\rho_3^2 - \rho_3) \frac{a_1'}{b} \right\},$$

and that

$$a_1'^2 = -108 b^2 (c_1^2 - c_2^3);$$

c_1 and c_2 having the determined values above referred to, namely

$$c_1 = -\tfrac{1}{54}(2a_1^3 - 9a_1 a_2 + 27 a_3), \quad c_2 = \tfrac{1}{9}(a_1^2 - 3a_2),$$

and ρ_3 being an imaginary cube-root of unity, but b and p_0, p_1, being any arbitrary rational functions of a_1, a_2, a_3, or even any arbitrary numeric constants; except that b must be different from 0, and that p_0, p_1 must not both together vanish. (In the formulæ of the earlier articles of this essay, these three last quantities had the following particular values,

$$b = \tfrac{1}{18}(\rho_3^2 - \rho_3), \quad p_0 = \tfrac{1}{3}, \quad p_1 = 0.)$$

By substituting for the cubic radical a_1'' the three unequal values a_1'', $\rho_3 a_1''$, $\rho_3^2 a_1''$, in the general expression, just now found, for x, we obtain separate and unequal expressions for the three separate roots x_1, x_2, x_3; these roots, and every rational function of them, may consequently be expressed as rational functions of the two radicals a_1' and a_1''; and therefore it is unnecessary and improper, in the present research, to introduce any other radical. But these two radicals a_1' and a_1'' are not essentially distinct from those which enter into the usual formulæ for the solution of a cubic equation: it is therefore impossible to discover any *new* irrational expression, finite and irreducible, for a root of the general cubic, *essentially distinct* from those which have long been known; and the only possible difference, with respect to the extracting of radicals, between any two methods of solution which both are free from all superfluous extractions, consists in the introduction of different square factors into that quantity or function f_1, of which, in each, the square root a_1' is to be calculated; or in the introduction of different cubic factors into that other quantity or function f_1', of which, in each method, it is requisite to calculate the cube-root a_1''. It is proper, however, to remember the remarks which have been made, in a foregoing article, respecting the *reducibility* of a certain expression, involving *two* cubic radicals a_1'' and a_2'', which is not uncommonly assigned for a root of the cubic equation.

[15.] But it is necessary to demonstrate some properties of rational functions of three variables, which have been employed in the foregoing investigation. And because it will be necessary to investigate afterwards some analogous properties of functions of four and five arbitrary quantities, it may be conducive to clearness and uniformity that we should begin with a few remarks respecting functions which involve two variables only.

Let $\mathrm{F}(x_\alpha, x_\beta)$ denote any arbitrary rational function of two arbitrary quantities x_1, x_2, arranged in either of their only two possible arrangements; so that the function F admits of the two following values

$$\mathrm{F}(x_1, x_2) \quad \text{and} \quad \mathrm{F}(x_2, x_1),$$

which for conciseness may be thus denoted,

$$(1, 2) \quad \text{and} \quad (2, 1).$$

These different *values* of the proposed function F may also be considered as being themselves two different *functions* of the same two quantities x_1, x_2 taken in some determined order; and may, in this view, be denoted thus,

$$\mathrm{F}_1(x_1, x_2) \quad \text{and} \quad \mathrm{F}_2(x_1, x_2),$$

or, more concisely,

$$(1, 2)_1 \quad \text{and} \quad (1, 2)_2:$$

they may also, on account of the mode in which they are formed from one common type $\mathrm{F}(x_\alpha, x_\beta)$ be said to be *syntypical functions.* For example, the two values,

$$ax_1 + ax_2 = (1, 2) = \mathrm{F}(x_1, x_2) = \mathrm{F}_1(x_1, x_2) = (1, 2)_1,$$

and

$$ax_2 + ax_1 = (2, 1) = \mathrm{F}(x_2, x_1) = \mathrm{F}_2(x_1, x_2) = (1, 2)_2,$$

of the function $ax_\alpha + bx_\beta$, may be considered as being two different but *syntypical* functions of the two variables x_1 and x_2. And again, in the same sense, the functions $\dfrac{x_1^2}{x_2}$ and $\dfrac{x_2^2}{x_1}$ are syntypical.

Now although, in general, two such syntypical functions, F_1 and F_2, are unconnected by any relation among themselves, on account of the independence of the two arbitrary quantities x_1 and x_2; yet, for some *particular forms* of the original or typical function F_1, they may become connected by some such relation, without any restriction being thereby imposed on those two arbitrary quantities. But all such relations may easily be investigated, with the help of the two general forms obtained in the thirteenth article, namely,

$$\mathrm{F}_1 = a + b(x_1 - x_2), \quad \mathrm{F}_2 = a - b(x_1 - x_2),$$

in which a and b are symmetric. For example, we see from these forms that the two syntypical functions F_1 and F_2 become equal, when they reduce themselves to the symmetric term or function a, but not in any other case; and that their squares are equal without their being equal themselves, if they are of the forms $\pm b(x_1 - x_2)$, but not otherwise. We see too, that we cannot suppose $\mathrm{F}_2 = \rho_3 \mathrm{F}_1$, without making a and b both vanish; and therefore that two syntypical functions of two arbitrary quantities cannot have equal cubes, if they be themselves unequal.

[16.] After these preliminary remarks respecting functions of two variables, let us now pass to functions of three; and accordingly let $\mathrm{F}(x_\alpha, x_\beta, x_\gamma)$, or more concisely (α, β, γ), denote any arbitrary rational function of any three arbitrary and independent quantities x_1, x_2, x_3, arranged in any arbitrary order. It is clear that this function F has in general six different values, namely,

$$(1,2,3), \quad (2,3,1), \quad (3,1,2), \quad (2,1,3), \quad (3,2,1), \quad (1,3,2),$$

or, in a more developed notation,

$$\mathrm{F}(x_1, x_2, x_3), \quad \ldots \quad \mathrm{F}(x_1, x_3, x_2),$$

corresponding to the six different possible arrangements of the three quantities on which it is supposed to depend; and that these six *values* of the function F may also be considered as six different but *syntypical functions* of the same three arbitrary quantities x_1, x_2, x_3, taken in some determined order; which functions may be thus denoted,

$$\mathrm{F}_1(x_1, x_2, x_3), \ldots \ \mathrm{F}_6(x_1, x_2, x_3),$$

or, more concisely,

$$(1,2,3)_1, \ldots \ (1,2,3)_6.$$

For example, the six following values,

$$\begin{aligned}
ax_1 + bx_2 + cx_3 &= (1,2,3) = \mathrm{F}(x_1, x_2, x_3),\\
ax_2 + bx_3 + cx_1 &= (2,3,1) = \mathrm{F}(x_2, x_3, x_1),\\
ax_3 + bx_1 + cx_2 &= (3,1,2) = \mathrm{F}(x_3, x_1, x_2),\\
ax_2 + bx_1 + cx_3 &= (2,1,3) = \mathrm{F}(x_2, x_1, x_3),\\
ax_3 + bx_2 + cx_1 &= (3,2,1) = \mathrm{F}(x_3, x_2, x_1),\\
ax_1 + bx_3 + cx_2 &= (1,3,2) = \mathrm{F}(x_1, x_3, x_2),
\end{aligned}$$

of the original or typical function

$$ax_\alpha + bx_\beta + cx_\gamma = \mathrm{F}(x_\alpha, x_\beta, x_\gamma),$$

may be considered as being six syntypical functions, F_1, F_2, F_3, F_4, F_5, F_6, of the three quantities x_1, x_2, x_3. Such also are the six following,

$$\frac{x_1}{x_2} + x_3, \quad \frac{x_2}{x_3} + x_1, \quad \frac{x_3}{x_1} + x_2, \quad \frac{x_2}{x_1} + x_3, \quad \frac{x_3}{x_2} + x_1, \quad \frac{x_1}{x_3} + x_2,$$

which are the values of the function $\dfrac{x_\alpha}{x_\beta} + x_\gamma$.

Now, in general, six such syntypical functions of three arbitrary quantities are all unequal among themselves; nor can any ratio or other relation between them be assigned, (except that very relation which constitutes them syntypical,) so long as the form of the function F, although it has been supposed to be rational, remains otherwise entirely undetermined. But, for some *particular forms* of this original or typical function $\mathrm{F}(x_\alpha, x_\beta, x_\gamma)$, relations may arise between the six syntypical functions $\mathrm{F}_1, \ldots \mathrm{F}_6$, without any restriction being thereby imposed on the three arbitrary quantities x_1, x_2, x_3; for example, the function F may be partially or wholly symmetric, and then the functions $\text{F}_1, \dots \text{F}_6$ will, some or all, be equal. And we are now to study the chief *functional conditions*, under which relations of this kind can arise. More precisely, we are to examine what are the conditions under which the number of the values of a rational function F of three variables, or of the square or cube of that function, can reduce itself below the number six, in consequence of two or more of the six syntypical functions $\text{F}_1, \dots \text{F}_6$, or of their squares or cubes, which are themselves syntypical, becoming equal to each other. And for this purpose we must first inquire into the conditions requisite in order that any *two* syntypical functions, or that any two values of F, may be equal.

[17.] If any two such values be denoted by the symbols

$$\text{F}(x_{\alpha_1}, x_{\beta_1}, x_{\gamma_1}), \quad \text{and} \quad \text{F}(x_{\alpha_2}, x_{\beta_2}, x_{\gamma_2}),$$

or, more concisely, by the following,

$$(\alpha_1, \beta_1, \gamma_1) \quad \text{and} \quad (\alpha_2, \beta_2, \gamma_2),$$

it is clear that in passing from the one to the other, and therefore in passing from some one arrangement to some other of the three indices α, β, γ, (which must themselves coincide, in some arrangement or other, with the numbers 1, 2, 3,) we must have changed some index, such as α, to some other, such as β, which must also have been changed, itself, either to α or to γ; this latter index γ remaining in the first case unaltered, but being changed to α in the second case. And, in whatever order the indices α_1, β_1, γ_1 may have coincided with α, β, γ, it is obvious that the function

$$\text{F}(x_{\alpha_1}, x_{\beta_1}, x_{\gamma_1}) \quad \text{or} \quad (\alpha_1, \beta_1, \gamma_1)$$

must coincide with the syntypical function

$$\text{F}_i(x_\alpha, x_\beta, x_\gamma) \quad \text{or} \quad (\alpha, \beta, \gamma)_i,$$

for some suitable index i, belonging to the system 1, 2, 3, 4, 5, 6; the equation

$$(\alpha_1, \beta_1, \gamma_1) = (\alpha_2, \beta_2, \gamma_2),$$

is therefore equal to one or other of the two following, namely, either

$$\text{1st}, \ \dots (\alpha, \beta, \gamma)_i = (\beta, \alpha, \gamma)_i,$$

or

$$\text{2nd}, \ \dots (\alpha, \beta, \gamma)_i = (\beta, \gamma, \alpha)_i.$$

In the first case, the function F_i is symmetric with respect to the two quantities x_α, x_β, and therefore involves them only by involving their sum and product, which may be thus expressed,

$$x_\alpha + x_\beta = -a_1 - x_\gamma, \quad x_\alpha x_\beta = a_2 + a_1 x_\gamma + x_\gamma^2,$$

a_1 and a_2 being symmetric functions of the three quantities x_1, x_2, x_3, namely, the following,

$$a_1 = -(x_1 + x_2 + x_3), \quad a_2 = x_1x_2 + x_1x_3 + x_2x_3;$$

so that if we put, for abridgment,

$$a_3 = -x_1x_2x_3,$$

the three quantities x_1, x_2, x_3 will be the three roots of the cubic equation

$$x^3 + a_1x^2 + a_2x + a_3 = 0.$$

In this case, therefore, we may consider F_i as being a rational function of the root x_γ alone, which function will however involve, in general, the coefficients a_1 and a_2; and we may put

$$\mathrm{F}_i(x_\alpha, x_\beta, x_\gamma) = \frac{\phi(x_\gamma)}{\chi(x_\gamma)} = \frac{\chi(x_\alpha) \,.\, \chi(x_\beta) \,.\, \phi(x_\gamma)}{\chi(x_\alpha) \,.\, \chi(x_\beta) \,.\, \chi(x_\gamma)} = \psi(x_\gamma),$$

ϕ, χ and ψ denoting here some rational and whole functions of x_γ, which may however involve rationally the coefficients of the foregoing cubic equation. And since it is unnecessary, on account of that equation, to retain in evidence the cube or any higher powers of x_γ, we may write simply

$$\mathrm{F}_i(x_\alpha, x_\beta, x_\gamma) = a + bx_\gamma + cx_\gamma^2,$$

a, b, c being here symmetric functions of the three quantities x_1, x_2, x_3: so that, in this case, the six syntypical functions, or values of the function F, reduce themselves to the three following

$$a + bx_1 + cx_1^2, \quad a + bx_2 + cx_2^2, \quad a + bx_3 + cx_3^2.$$

Nor can these three reduce themselves to any smaller number, without their all becoming equal and symmetric, by the vanishing of b and c.

In the second case, the form of F_i being such that

$$(\alpha, \beta, \gamma)_i = (\beta, \gamma, \alpha)_i,$$

it must also be such that

$$(\beta, \gamma, \alpha)_i = (\gamma, \alpha, \beta)_i;$$

for the same reason we must have

$$(\beta, \alpha, \gamma)_i = (\alpha, \gamma, \beta)_i = (\gamma, \beta, \alpha)_i,$$

so that the function changes when any two of the three indices are interchanged, but returns to its former value when any two are interchanged again; from which it results that the two following combinations

$$(\alpha, \beta, \gamma)_i + (\beta, \alpha, \gamma)_i \quad \text{and} \quad \frac{(\alpha, \beta, \gamma)_i - (\beta, \alpha, \gamma)_i}{(x_\alpha - x_\beta)(x_\alpha - x_\gamma)(x_\beta - x_\gamma)}$$

remain unchanged, after all interchanges of the indices, and are therefore symmetric functions, such as $2a$ and $2b$, of the three quantities x_1, x_2, x_3: so that we may write

$$\mathrm{F}_i(x_\alpha, x_\beta, x_\gamma) = (\alpha, \beta, \gamma)_i = a + b(x_\alpha - x_\beta)(x_\alpha - x_\gamma)(x_\beta - x_\gamma);$$

and consequently the six syntypical functions, or values of the function F, reduce themselves in this case to the two following,

$$a \pm b(x_1 - x_2)(x_1 - x_3)(x_2 - x_3),$$

in which a and b are symmetric. It is evident that any farther diminution of the number of values of F, conducts, in this case also, to the one-valued or symmetric function a.

Combining the foregoing results, we see that if an unsymmetric rational function of three arbitrary quantities have fewer than six values, it must be reducible either to the two-valued form

$$a + b(x_1 - x_2)(x_1 - x_3)(x_2 - x_3),$$

or to the three-valued form

$$a + bx + cx^2.$$

[18.] It is possible, however, that some analogous but different reduction may cause either—I. the square, or II. the cube of a function F of three variables, to have a smaller number of values than the function F itself. But, for this purpose, it is necessary that we should now have a relation of one or other of the two forms following, namely, either

$$\text{I.}\dots \quad (\alpha_2, \beta_2, \gamma_2) = -(\alpha_1, \beta_1, \gamma_1)$$

or

$$\text{II.}\dots \quad (\alpha_2, \beta_2, \gamma_2) = \rho_3(\alpha_1, \beta_1, \gamma_1)$$

(ρ_3 denoting, as above, an imaginary cube root of unity,) instead of the old functional relation $(\alpha_2, \beta_2, \gamma_2) = (\alpha_1, \beta_1, \gamma_1)$. And as we found ourselves permitted, before, to change that old relation to one or other of these two,

$$\text{1st,} \quad (\beta, \alpha, \gamma)_i = (\alpha, \beta, \gamma)_i; \quad \text{2nd,} \quad (\beta, \gamma, \alpha)_i = (\alpha, \beta, \gamma)_i;$$

so are we now allowed to change the two new relations to the four following:

$$\begin{array}{llll} \text{I. 1,} \dots & (\beta, \alpha, \gamma)_i = -(\alpha, \beta, \gamma)_i; & \text{I. 2,} \dots & (\beta, \gamma, \alpha)_i = -(\alpha, \beta, \gamma)_i; \\ \text{II. 1,} \dots & (\beta, \alpha, \gamma)_i = \rho_3(\alpha, \beta, \gamma)_i; & \text{II. 2,} \dots & (\beta, \gamma, \alpha)_i = \rho_3(\alpha, \beta, \gamma)_i; \end{array}$$

the relation (I.) admitting of being changed to one or other of the two marked (I. 1) and (I. 2); and the relation (II.) admitting, in like manner, of being changed either to (II. 1) or to (II. 2). But the relations (I. 2) and (II. 1) conduct only to evanescent functions, because (I. 2) gives

$$(\gamma, \alpha, \beta)_i = -(\beta, \gamma, \alpha)_i = +(\alpha, \beta, \gamma)_i, \quad (\alpha, \beta, \gamma)_i = -(\gamma, \alpha, \beta)_i = -(\alpha, \beta, \gamma)_i,$$

and (II. 1) gives

$$(\alpha, \beta, \gamma)_i = \rho_3(\beta, \alpha, \gamma)_i = \rho_3^2(\alpha, \beta, \gamma)_i :$$

we may therefore confine our attention to the other two relations. Of these (I. 1) requires that the function $\dfrac{(\alpha, \beta, \gamma)_i}{(x_\alpha - x_\beta)}$ should not change its value when x_α and x_β are interchanged, and consequently, by what was shown above, that it should be reducible to the form $a + bx_\gamma + cx_\gamma^2$; in this case, therefore, we have the expression,

$$(\alpha, \beta, \gamma)_i = \mathrm{F}_i(x_\alpha, x_\beta, x_\gamma) = (x_\alpha - x_\beta)(a + bx_\gamma + cx_\gamma^2),$$

the coefficients a, b, c, being symmetric functions of x_1, x_2, x_3. Accordingly the square of this function F_i admits in general of three values only, while the function is itself in general six-valued; because the square of the factor $x_\alpha - x_\beta$, but not that factor itself, can be expressed as a rational function of x_γ, and of the quantities a_1, a_2, a_3, which are symmetric relatively to x_1, x_2, x_3. It may even happen that the function itself shall have only two values, and that its square shall be symmetric, namely, by the factor $a + bx_\gamma + cx_\gamma^2$ being reducible to the form $b(x_\alpha - x_\gamma)(x_\beta - x_\gamma)$, in which the coefficient b is some new symmetric function; but the results of the last article enable us to see that the functions thus obtained, namely, those of the form

$$b(x_\alpha - x_\beta)(x_\alpha - x_\gamma)(x_\beta - x_\gamma),$$

or more simply of the form

$$b(x_1 - x_2)(x_1 - x_3)(x_2 - x_3),$$

are the only two-valued functions of three variables which have symmetric squares: they enable us also to see easily that the square of a three-valued function of three variables is always itself three-valued. It remains, then, only to consider the relations (II. 2); which requires that the function

$$\frac{(\alpha, \beta, \gamma)_i}{x_\alpha + \rho_3^2 x_\beta + \rho_3 x_\gamma}$$

should be of the two-valued form $a + b(x_\alpha - x_\beta)(x_\alpha - x_\gamma)(x_\beta - x_\gamma)$; because, if we denote it by $\phi(x_\alpha, x_\beta, x_\gamma)$, we have

$$\phi(x_\alpha, x_\beta, x_\gamma) = \phi(x_\beta, x_\gamma, x_\alpha) = \phi(x_\gamma, x_\alpha, x_\beta),$$

and

$$\phi(x_\beta, x_\alpha, x_\gamma) = \phi(x_\alpha, x_\gamma, x_\beta) = \phi(x_\gamma, x_\beta, x_\alpha);$$

we have, therefore, in this case,

$$\begin{aligned}(\alpha, \beta, \gamma)_i &= \mathrm{F}_i(x_\alpha, x_\beta, x_\gamma) \\ &= \{a + b(x_\alpha - x_\beta)(x_\alpha - x_\gamma)(x_\beta - x_\gamma)\}(x_\alpha + \rho_3^2 x_\beta + \rho_3 x_\gamma),\end{aligned}$$

a and b being symmetric coefficients, which must not both together vanish; and accordingly we find, *à posteriori*, that whereas this function F_i has always itself six values, its cube has only two. The foregoing analysis shows at the same time, that if an unsymmetric function of three variables have fewer than six values, its cube cannot have fewer values than itself; and accordingly it is easy to see that the cubes of those two-valued and three-valued functions, which were assigned in the last article, are themselves two-valued and three-valued. In fact, the passage from any one to any other of the values of any such (two-valued or three-valued) function, may be performed by interchanging some two of the three quantities x_1, x_2, x_3; and if such interchange could have the effect of multiplying the function by an imaginary cube-root of unity, ρ_3, another interchange of the same two quantities would multiply again by the same factor ρ_3; and therefore the two interchanges combined would multiply by ρ_3^2, which is a factor different from unity, although any two such successive interchanges of any two quantities x_α, x_β, ought to make no change in the function. If, then, a rational function of three arbitrary quantities have a symmetric cube, it must be itself symmetric.

The form of that six-valued function of three variables which has a two-valued cube, may also be thus deduced, from the functional relation (II. 2). Omitting for simplicity, the lower index i, which is not essential to the reasoning, we find, by that relation,

$$(\beta,\gamma,\alpha) = \rho_3(\alpha,\beta,\gamma); \quad (\gamma,\alpha,\beta) = \rho_3^2(\alpha,\beta,\gamma);$$

$$(\gamma,\beta,\alpha) = \rho_3(\alpha,\gamma,\beta); \quad (\beta,\alpha,\gamma) = \rho_3^2(\alpha,\gamma,\beta);$$

so that

$$(\alpha,\beta,\gamma)\,.\,(\alpha,\gamma,\beta) = (\beta,\gamma,\alpha)(\beta,\alpha,\gamma) = (\gamma,\alpha,\beta)(\gamma,\beta,\alpha) = e,$$

this product e being some symmetric function; at the same time, the sum $(\alpha,\beta,\gamma)+(\alpha,\gamma,\beta)$ is a three-valued function y_α, which may be put under the form

$$y_\alpha = a + bx_\alpha + cx_\alpha^2,$$

a, b, and c being symmetric, and b and c being obliged not both to vanish. Attending therefore to that cubic equation of which x_α, x_β, x_γ are the roots, we have

$$y_\alpha^2 = a^{(2)} + b^{(2)}x_\alpha + c^{(2)}x_\alpha^2,$$

$a^{(2)}$, $b^{(2)}$, and $c^{(2)}$ denoting here some symmetric functions, and c, $c^{(2)}$ being obliged not both to vanish; and consequently, by eliminating x_α^2, we obtain an equation of the form

$$(bc^{(2)} - cb^{(2)})x_\alpha = ca^{(2)} - ac^{(2)} + c^{(2)}y_\alpha - cy_\alpha^2,$$

in which the coefficients of y_α and y_α^2 cannot both vanish, and in which therefore the coefficient of x_α cannot vanish, because the three-valued function y_α must not be a root of any equation with symmetric coefficients, below the third degree; we have therefore an expression of the form

$$x_\alpha = p + qy_\alpha + ry_\alpha^2,$$

in which p, q, r are symmetric, and q and r do not both vanish. But

$$y_\alpha = (\alpha,\beta,\gamma) + (\alpha,\gamma,\beta) = (\alpha,\beta,\gamma) + \frac{e}{(\alpha,\beta,\gamma)};$$

and the cube of (α, β, γ) is a two-valued function; therefore

$$x_\alpha = p' + q'(\alpha, \beta, \gamma) + r'(\alpha, \beta, \gamma)^2,$$

the functions p', q', r' being either symmetric or two-valued, and consequently undergoing no change, when we pass successively from the first to the second, or from the second to the third, of the three functions (α, β, γ), (β, γ, α), (γ, α, β), by changing at each passage x_α to x_β, x_β to x_γ, and x_γ to x_α; and we have seen that these three last-mentioned functions bear to each other the same ratios as the three cube-roots of unity 1, ρ_3, ρ_3^2; we have therefore

$$x_\beta = p' + q'\rho_3(\alpha, \beta, \gamma) + r'\rho_3^2(\alpha, \beta, \gamma)^2, \quad x_\gamma = p' + q'\rho_3^2(\alpha, \beta, \gamma) + r'\rho_3(\alpha, \beta, \gamma)^2;$$

and thus, finally, the six-valued function which has a two-valued cube is found anew to be expressible as follows,

$$(\alpha, \beta, \gamma) = \frac{1}{3q'}(x_\alpha + \rho_3^2 x_\beta + \rho_3 x_\gamma);$$

in which the coefficient $\dfrac{1}{3q'}$ is a two-valued function, of the form

$$\frac{1}{3q'} = a + b(x_1 - x_2)(x_1 - x_3)(x_2 - x_3),$$

a, b, denoting here some new symmetric functions.

The theorems obtained incidentally in this last discussion supply us also with another mode of proving that the cube of a three-valued function of three arbitrary quantities must be itself three-valued: for if we should suppose $y_\beta = \rho_3 y_\alpha$, and consequently $y_\gamma = \rho_3 y_\beta = \rho_3^2 y_\alpha$, in which $y_\alpha = a + bx_\alpha + cx_\alpha^2$, and b and c do not both vanish, we should then have relations of the forms

$$\begin{aligned} x_\alpha &= p + qy_\alpha + ry_\alpha^2, \\ x_\beta &= p + q\rho_3 y_\alpha + r\rho_3^2 y_\alpha^2, \\ x_\gamma &= p + q\rho_3^2 y_\alpha + r\rho_3 y_\alpha^2; \end{aligned}$$

but these would require that we should have the equation

$$x_\alpha + \rho_3^2 x_\beta + \rho_3 x_\gamma = 3q \,.\, y_\alpha,$$

a condition which it is impossible to fulfil, because the first member has six values, and the second only three.

[19.] The discussion of the forms of functions of four variables may now be conducted more briefly, than would have been consistent with clearness, if we had not already treated so fully of functions in which the number of variables is less than four.

Let x_1, x_2, x_3, x_4 be any four arbitrary quantities, or roots of the general biquadratic,

$$x^4 + a_1 x^3 + a_2 x^2 + a_3 x + a_4 = 0;$$

and let $\text{F}(x_1, x_2, x_3, x_4)$, or, more concisely $(1, 2, 3, 4)$, denote any rational function of them. By altering the arrangement of these four roots, we shall in general obtain twenty-four different but *syntypical* functions; of which each, according to the analogy of the foregoing notation, may be denoted by any one of the four following symbols:

$$(\alpha, \beta, \gamma, \delta) = \text{F}(x_\alpha, x_\beta, x_\gamma, x_\delta) = (1, 2, 3, 4)_i = \text{F}_i(x_1, x_2, x_3, x_4).$$

In passing from any one to any other of these twenty-four syntypical functions $\text{F}_1, \ldots \text{F}_{24}$, by a change of arrangement of the four roots, some one of these roots, such as the first in order, must be changed to some other, such as the second; and this second must at the same time be changed either to the first or to a different root, such as the third; while, in the former case, the third and fourth roots may either be interchanged among themselves or not; and, in the latter case, the third root may be changed either to the first or to the fourth. We have therefore four and only four distinct sorts of changes of arrangement, which may be typified by the passages from the function $(\alpha, \beta, \gamma, \delta)$ to the four following:

$$\text{I.} \ \ldots (\beta, \alpha, \gamma, \delta); \quad \text{II.} \ \ldots (\beta, \alpha, \delta, \gamma); \quad \text{III.} \ \ldots (\beta, \gamma, \alpha, \delta); \quad \text{IV.} \ \ldots (\beta, \gamma, \delta, \alpha);$$

and may be denoted by the four characteristics

$$\nabla_1, \quad \nabla_2, \quad , \nabla_3, \quad \nabla_4;$$

or more fully by the following,

$$\overset{a,b}{\nabla}_1, \quad \overset{a,b}{\nabla}_2, \quad \overset{a,b,c}{\nabla}_3, \quad \overset{a,b,c}{\nabla}_4;$$

$\overset{a,b}{\nabla}_1$ implying, when prefixed to any function $(\alpha, \beta, \gamma, \delta)$, that we are to interchange the a^{th} and b^{th} of the roots on which it depends; $\overset{a,b}{\nabla}_2$, that we are to interchange among themselves, not only the a^{th} and b^{th}, but also that c^{th} and d^{th}; $\overset{a,b,c}{\nabla}_3$, that we are to interchange the a^{th} to the b^{th}, the b^{th} to the c^{th}, and the c^{th} to the a^{th}; namely, by putting that which had been b^{th} in the place of that which had been a^{th}, and so on; and finally $\overset{a,b,c}{\nabla}_4$, that the a^{th} is to be changed to the b^{th}, the b^{th} to the c^{th}, the c^{th} to the d^{th}, and the d^{th} to the a^{th}; so that we have, in this notation,

$$\text{I.} \ \ldots \overset{1,2}{\nabla}_1(\alpha, \beta, \gamma, \delta) = (\beta, \alpha, \gamma, \delta);$$

$$\text{II.} \ \ldots \overset{1,2}{\nabla}_2(\alpha, \beta, \gamma, \delta) = (\beta, \alpha, \delta, \gamma);$$

$$\text{III.} \ \ldots \overset{1,2,3}{\nabla}_3(\alpha, \beta, \gamma, \delta) = (\beta, \gamma, \alpha, \delta);$$

$$\text{IV.} \ \ldots \overset{1,2,3}{\nabla}_4(\alpha, \beta, \gamma, \delta) = (\beta, \gamma, \delta, \alpha).$$

The first sort of change may be called, altering in a *simple binary cycle*; the second, in a *double binary cycle*; the third, in a *ternary*; and the fourth, in a *quaternary* cycle. And every possible equation,

$$(\alpha_2, \beta_2, \gamma_2, \delta_2) = (\alpha_1, \beta_1, \gamma_1, \delta_1),$$

between any two of the twenty-four syntypical functions F_i may be denoted by one or other of the four following symbolic forms, in each of which the two members may be conceived to be prefixed to a function such as $(\alpha_1, \beta_1, \gamma_1, \delta_1)$:

$$\text{I.} \ \ldots \ \overset{a,b}{\nabla}_1 = 1; \quad \text{II.} \ \ldots \ \overset{a,b}{\nabla}_2 = 1; \quad \text{III.} \ \ldots \ \overset{a,b,c}{\nabla}_3 = 1; \quad \text{IV.} \ \ldots \ \overset{a,b,c}{\nabla}_4 = 1;$$

or, without any loss of generality, by one of the four following, in each of which the two members are conceived to be prefixed to a function such as $(\alpha, \beta, \gamma, \delta)_i$:

$$\text{I.} \ \ldots \ \overset{1,2}{\nabla}_1 = 1; \quad \text{II.} \ \ldots \ \overset{1,2}{\nabla}_2 = 1; \quad \text{III.} \ \ldots \ \overset{1,2,3}{\nabla}_3 = 1; \quad \text{IV.} \ \ldots \ \overset{1,2,3}{\nabla}_4 = 1;$$

the I$^{\text{st}}$ and II$^{\text{nd}}$ suppositions conducting to twelve-valued functions, the III$^{\text{rd}}$ to an eight-valued, and the IV$^{\text{th}}$ to a six-valued function; while every possible pair of equations between any three of the same twenty-four syntypical functions, if it be not included in a single equation of this last set, may be put under one or other of the six following forms:

$$\begin{array}{llll}
\text{(I. I.)} & \ldots \ \overset{1,2}{\nabla}_1 = 1, \ \ \overset{1,3}{\nabla}_1 = 1; & \text{(I. I.)}' & \ldots \ \overset{1,2}{\nabla}_1 = 1, \ \ \overset{3,4}{\nabla}_1 = 1; \\
\text{(I. II.)} & \ldots \ \overset{1,2}{\nabla}_1 = 1, \ \ \overset{1,3}{\nabla}_2 = 1; & \text{(I. III.)} & \ldots \ \overset{1,2}{\nabla}_1 = 1, \ \ \overset{2,3,4}{\nabla}_3 = 1; \\
\text{(II. II.)} & \ldots \ \overset{1,2}{\nabla}_2 = 1, \ \ \overset{1,3}{\nabla}_2 = 1; & \text{(II. III.)} & \ldots \ \overset{1,2}{\nabla}_2 = 1, \ \ \overset{1,2,3}{\nabla}_3 = 1;
\end{array}$$

which conduct respectively to functions with four, six, three, one, six and two values; nor can any form of condition, essentially distinct from all the ten last mentioned, be obtained by supposing any three or more equations to exist between the twenty-four functions F_i.

A little attention will not fail to evince the justice of this enumeration of the conditions under which a rational function of four arbitrary variables can have fewer than twenty-four values: yet it may not be useless to remark, as connected with this inquiry, that, in virtue of the notation here employed, the supposition $\overset{a,b}{\nabla}_1 = 1$ involves the supposition $\overset{b,a}{\nabla}_1 = 1$; the supposition $\overset{a,b}{\nabla}_2 = 1$ involves the suppositions $\overset{b,a}{\nabla}_2 = 1$, $\overset{c,d}{\nabla}_2 = 1$, $\overset{d,c}{\nabla}_2 = 1$; $\overset{a,b,c}{\nabla}_3 = 1$ involves $\overset{b,c,a}{\nabla}_3 = 1$, $\overset{c,a,b}{\nabla}_3 = 1$, $\overset{a,c,b}{\nabla}_3 = 1$, $\overset{c,b,a}{\nabla}_3 = 1$, $\overset{b,a,c}{\nabla}_3 = 1$; $\overset{a,b,c}{\nabla}_4 = 1$ involves $\overset{b,c,d}{\nabla}_4 = 1$, $\overset{c,d,a}{\nabla}_4 = 1$, $\overset{d,a,b}{\nabla}_4 = 1$, $\overset{a,c}{\nabla}_2 = 1$, $\overset{a,d,c}{\nabla}_4 = 1$, $\overset{d,c,b}{\nabla}_4 = 1$, $\overset{c,b,a}{\nabla}_4 = 1$, $\overset{b,a,d}{\nabla}_4 = 1$; while the system $\overset{a,b}{\nabla}_1 = 1$, $\overset{a,b}{\nabla}_2 = 1$, is equivalent to the system $\overset{a,b}{\nabla}_1 = 1$, $\overset{c,d}{\nabla}_1 = 1$; $\overset{a,b}{\nabla}_1 = 1$, $\overset{a,b,c}{\nabla}_3 = 1$, to $\overset{a,b}{\nabla}_1 = 1$, $\overset{a,c}{\nabla}_1 = 1$; $\overset{a,b}{\nabla}_1 = 1$, $\overset{a,b,c}{\nabla}_4 = 1$, to $\overset{a,b}{\nabla}_1 = 1$, $\overset{b,c,d}{\nabla}_3 = 1$; $\overset{a,b}{\nabla}_1 = 1$, $\overset{a,c,b}{\nabla}_4 = 1$, to $\overset{a,b}{\nabla}_1 = 1$, $\overset{a,c}{\nabla}_2 = 1$; $\overset{a,b}{\nabla}_2 = 1$, $\overset{a,b,c}{\nabla}_4 = 1$, to $\overset{a,b}{\nabla}_2 = 1$, $\overset{b,d}{\nabla}_1 = 1$; $\overset{a,b,c}{\nabla}_3 = 1$, $\overset{b,c,d}{\nabla}_3 = 1$, to $\overset{a,b}{\nabla}_2 = 1$, $\overset{a,b,c}{\nabla}_3 = 1$; $\overset{a,b,c}{\nabla}_3 = 1$, $\overset{a,b,c}{\nabla}_4 = 1$, to $\overset{a,b,c}{\nabla}_3 = 1$, $\overset{c,d}{\nabla}_1 = 1$; and $\overset{a,b,c}{\nabla}_4 = 1$, $\overset{a,c,b}{\nabla}_4 = 1$, to $\overset{a,b,c}{\nabla}_3 = 1$, $\overset{a,b,c}{\nabla}_4 = 1$: analogous equivalences also holding good for other systems of analogous conditions.

Let us now consider more closely the effects of the ten different suppositions (I.),... (II. III.).

In the case (I.), the function F is symmetric relatively to some two roots x_α, x_β, and may be put under the form of a rational function of two others, x_γ, x_δ, or simply of their difference,

$$(\text{I.}) \ldots F = \phi(x_\gamma - x_\delta);$$

it being understood that this function ϕ may involve the coefficients a_1, a_2, a_3, a_4, which are symmetric relatively to x_1, x_2, x_3, x_4: because it is in general possible to determine rationally any two roots x_γ, x_δ, of an equation of any given degree, when their difference $x_\gamma - x_\delta$ is given.

In the case (II.), we may interchange some two roots, x_α, x_β, if we at the same time interchange the two others; and the function may be put under the form

$$(\text{II.}) \ldots F = \phi(x_\alpha + x_\beta - x_\gamma - x_\delta, \overline{x_\alpha - x_\beta} \,.\, \overline{x_\gamma - x_\delta});$$

because any rational function of the four roots may be considered as a rational function of the four combinations

$$x_\alpha + x_\beta, \quad x_\alpha - x_\beta, \quad x_\gamma + x_\delta, \quad x_\gamma - x_\delta,$$

or of the four following,

$$x_\alpha + x_\beta + x_\gamma + x_\delta, \quad x_\alpha + x_\beta - x_\gamma - x_\delta, \quad x_\alpha - x_\beta, \quad \overline{x_\alpha - x_\beta} \,.\, \overline{x_\gamma - x_\delta};$$

of which the first may be omitted, as symmetric, and the third as being here obliged to enter only by its square, which square $(x_\alpha - x_\beta)^2$ is expressible as a rational function of $x_\alpha + x_\beta - x_\gamma - x_\delta$, involving also the symmetric coefficients a_1, a_2, a_3, which are allowed to enter in any manner into ϕ.

In the case (III.), some three roots x_α, x_β, x_γ, may all be interchanged, the fourth root remaining unaltered; and, on account of what has been shown respecting functions of three variables, we may write

$$(\text{III.}) \ldots F = \phi(x_\delta) + (x_\alpha - x_\beta)(x_\alpha - x_\gamma)(x_\beta - x_\gamma)\psi(x_\delta),$$

the function ψ (as well as ϕ) being rational.

In the case (IV.), we may change x_α to x_β, if we at the same time change x_β to x_γ, x_γ to x_δ, and x_δ to x_α; and the function F is of the form

$$(\text{IV.}) \ldots F = \phi(\overline{x_\alpha - x_\beta + x_\gamma - x_\delta} \,.\, \overline{x_\alpha - x_\gamma} \,.\, \overline{x_\beta - x_\delta});$$

because the condition $\overset{1,2,3}{\nabla_4} = 1$ involves the condition $\overset{1,3}{\nabla_2} = 1$, and consequently the present function F must be rational relatively to the two combinations

$$x_\alpha + x_\gamma - x_\beta - x_\delta \quad \text{and} \quad \overline{x_\alpha - x_\gamma} \,.\, \overline{x_\beta - x_\delta};$$

or relatively to the two following,

$$x_\alpha - x_\beta + x_\gamma - x_\delta \quad \text{and} \quad \overline{x_\alpha - x_\beta + x_\gamma - x_\delta} \,.\, \overline{x_\alpha - x_\gamma} \,.\, \overline{x_\beta - x_\delta};$$

but of these two last-mentioned combinations, the former alone changes, and it changes in its sign alone, when the operation $\overset{1,2,3}{\nabla_4}$ performed, so that it can enter only by its square; which square $(x_\alpha - x_\beta + x_\gamma - x_\delta)^2$ can be expressed as a rational function of the product

$$(x_\alpha - x_\beta + x_\gamma - x_\delta)(x_\alpha - x_\gamma)(x_\beta - x_\delta),$$

and of those symmetric coefficients which may enter in any manner into ϕ.

By similar reasonings it appears, that in the six other cases (I. I.) ... (II. III.), we have, respectively, the six following forms for F:

$$(\text{I. I.}) \ldots F = \phi(x_\delta) = a + bx_\delta + cx_\delta^2 + dx_\delta^3;$$

$$(\text{I. I.})' \ldots F = \phi(x_\alpha + x_\beta - x_\gamma - x_\delta);$$

$$(\text{I. II.}) \ldots F = \phi(x_\alpha x_\beta + x_\gamma x_\delta) = a + b(x_\alpha x_\beta + x_\gamma x_\delta) + c(x_\alpha x_\beta + x_\gamma x_\delta)^2;$$

$$(\text{I. III.}) \ldots F = a;$$

$$(\text{II. II.}) \ldots F = \phi(\overline{x_\alpha - x_\beta} \,.\, \overline{x_\gamma - x_\delta});$$

$$(\text{II. III.}) \ldots F = a + b(x_\alpha - x_\beta)(x_\alpha - x_\gamma)(x_\alpha - x_\delta)(x_\beta - x_\gamma)(x_\beta - x_\delta)(x_\gamma - x_\delta).$$

To one or other of the ten forms last determined, may therefore be reduced every rational function of four arbitrary quantities, which has fewer than twenty-four values. And although the functions (I. I.)′ and (II. II.) are six-valued, as well as the function (IV.), yet these three functions are all in general distinct from one another; the function (IV.) being one which does not change its value, when the four roots x_α, x_β, x_γ, x_δ are all changed in some one *quaternary cycle*, but the function (I. I.)′ being one which allows *either or both* of *some two pairs* x_α, x_β and x_γ, x_δ to have its two roots interchanged, and the function (II. II.) being characterized by its allowing *any two roots* to be interchanged, if *the other two roots* be interchanged at the same time. It may be useful also to observe, that the three-valued function (I. II.) belongs, as a particular case, to each of these three six-valued forms, and may easily be deduced from the form (I. I.)′, as follows:

$$\text{F} = \psi(x_\alpha + x_\beta - x_\gamma - x_\delta) = \psi(x_\gamma + x_\delta - x_\alpha - x_\beta) = \chi\overline{(x_\alpha + x_\beta - x_\gamma - x_\delta)^2} = \phi(x_\alpha x_\beta + x_\gamma x_\delta).$$

Attending next to conditions of the forms

$$\nabla = -1, \quad \nabla = \rho_3,$$

instead of attending only to conditions of the form

$$\nabla = 1,$$

we discover the forms which a rational function of four arbitrary variables must have, in order that its square or cube may have fewer values than itself; which functional forms are the following:

The general twenty-four-valued function F will have its square twelve-valued, if it be either of the form

$$\text{F} = (x_\alpha - x_\beta) \,.\, \psi(x_\gamma - x_\delta),$$

or of this other form

$$\text{F} = (x_\alpha - x_\beta) \,.\, \psi(x_\alpha + x_\beta - x_\gamma - x_\delta, \overline{x_\alpha - x_\beta} \,.\, \overline{x_\gamma - x_\delta}).$$

The same general or twenty-four-valued function will have an eight-valued cube, if it be of the form

$$\text{F} = \{\phi(x_\delta) + (x_\alpha - x_\beta)(x_\alpha - x_\gamma)(x_\beta - x_\gamma)\psi(x_\delta)\}(x_\alpha + \rho_3^2 x_\beta + \rho_3 x_\gamma),$$

ρ_3 being, as before, an imaginary cube-root of unity. The twelve-valued function (I.) will have a six-valued square, if it be reducible to the form

$$\text{F} = (x_\gamma - x_\delta) \,.\, \psi(x_\alpha + x_\beta - x_\gamma - x_\delta).$$

The twelve-valued function (II.) will have a six-valued square, if it be of the form

$$\text{F} = (x_\alpha + x_\beta - x_\gamma - x_\delta) \,.\, \psi(\overline{x_\alpha - x_\beta} \,.\, \overline{x_\gamma - x_\delta}),$$

or of the form

$$\text{F} = (x_\alpha - x_\beta)(x_\gamma - x_\delta) \,.\, \psi(x_\alpha + x_\beta - x_\gamma - x_\delta),$$

or of the form

$$\text{F} = (x_\alpha + x_\beta - x_\gamma - x_\delta) \,.\, \psi(\overline{x_\alpha + x_\beta - x_\gamma - x_\delta} \,.\, \overline{x_\alpha - x_\beta} \,.\, \overline{x_\gamma - x_\delta}).$$

The eight-valued function (III.) will have its square four-valued, if it be of the form

$$\text{F} = (x_\alpha - x_\beta)(x_\alpha - x_\gamma)(x_\beta - x_\gamma)\psi(x_\delta).$$

The six-valued functions (IV.), (I. I.)$'$, (II. II.), will have their squares three-valued, if they be reducible, respectively, to the forms,

$$\text{F} = (x_\alpha - x_\beta + x_\gamma - x_\delta)(x_\alpha - x_\gamma)(x_\beta - x_\delta) \,.\, \psi(x_\alpha x_\gamma + x_\beta x_\delta),$$

$$\text{F} = (x_\alpha + x_\beta - x_\gamma - x_\delta) \,.\, \psi(x_\alpha x_\beta + x_\gamma x_\delta),$$

$$\text{F} = (x_\alpha - x_\beta)(x_\gamma - x_\delta) \,.\, \psi(x_\alpha x_\beta + x_\gamma x_\delta);$$

and the last-mentioned six-valued function, (II. II.), will have its cube two-valued, if it be reducible to the form

$$F = \{a + b(x_\alpha - x_\beta)(x_\alpha - x_\gamma)(x_\alpha - x_\delta)(x_\beta - x_\gamma)(x_\beta - x_\delta)(x_\gamma - x_\delta)\}$$
$$\times\{x_\alpha x_\beta + x_\gamma x_\delta + \rho_3^2(x_\alpha x_\gamma + x_\beta x_\delta) + \rho_3(x_\alpha x_\delta + x_\beta x_\gamma)\},$$

ρ_3 being still an imaginary cube-root of unity. And the square of the two-valued function (II. III.) will be symmetric, if it be of the form

$$\text{F} = b(x_\alpha - x_\beta)(x_\alpha - x_\gamma)(x_\alpha - x_\delta)(x_\beta - x_\gamma)(x_\beta - x_\delta)(x_\gamma - x_\delta).$$

But there exists no other case of reduction essentially distinct from these, in which the number of values of the square or cube of a rational function of four independent variables is less than the number of values of that function itself. Some steps, indeed, have been for brevity omitted, which would be requisite for the full statement of a formal demonstration of all the foregoing theorems; but these omitted steps will easily occur to any one, who has considered with attention the investigation of the properties of rational functions of three variables, given in the two preceding articles.

[20.] The foregoing theorem respecting functions of four variables being admitted, let us now proceed to apply them to the *à priori* investigation of all possible expressions, finite and irreducible, of the form $b^{(m)}$, for a root x of the general biquadratic equation already often referred to, namely,

$$x^4 + a_1 x^3 + a_2 x^2 + a_3 x + a_4 = 0.$$

It is evident in the first place that we cannot express any such root x as a rational function of the coefficients a_1, a_2, a_3, a_4, because these are symmetric relatively to the four roots x_1, x_2, x_3, x_4, and a symmetric function of four arbitrary and independent quantities cannot be equal to an unsymmetric function of them; we must therefore suppose that m in $b^{(m)}$ is greater than 0, or, in other words, that the function $b^{(m)}$ is irrational, with respect to the quantities a_1, a_2, a_3, a_4, if any expression of the required kind can be found at all for x. On the other hand, the general theorem of ABEL shows that if any such expression exist, it must be composed of some finite combination of quadratic and cubic radicals, together with rational functions; because 2 and 3 are the only prime divisors of the product $24 = 1\,.\,2\,.\,3\,.\,4$. And the first and only radical of the first order in $b^{(m)}$, must be a square-root, of the form

$$\begin{aligned} a'_1 &= b(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_3 - x_4) \\ &= \sqrt{-442368\,.\,b^2\,.\,(e_1^2 - e_2^3)} = \sqrt{-2^{14}\,.\,3^3\,.\,b^2\,.\,(e_1^2 - e_2^3)}, \end{aligned}$$

b being some symmetric function of x_1, x_2, x_3, x_4, and e_1, e_2 having the same meanings here as in the second article; because no rational and unsymmetric function of four arbitrary quantities x_1, x_2, x_3, x_4, has a prime power symmetric, except either this function a'_1, or else some other function a'_2 which may be deduced from it by a multiplication such as the following, $a'_2 = \dfrac{c}{b} a'_1$. But a two-valued expression of the form $f'_1 = b_0 + b_1 a'_1$ cannot represent a four-valued function, such as x; we must therefore suppose that the sought expression $b^{(m)}$ contains a radical a''_1 of the second order, and this must be a cube-root, of the form

$$a''_1 = (p_0 + p_1 a'_1)(u_1 + \rho_3^2 u_2 + \rho_3 u_3) = \sqrt[3]{(b_0 + b_1 a'_1)};$$

in which ρ_3 is, as before, an imaginary cube-root of unity; p_0, p_1, b_0, b_1 are symmetric relatively to x_1, x_2, x_3, x_4, or rational relatively to a_1, a_2, a_3, a_4;

$$u_1 = x_1 x_2 + x_3 x_4, \quad u_2 = x_1 x_3 + x_2 x_4, \quad u_3 = x_1 x_4 + x_2 x_3;$$

and

$$b_0 + b_1 a'_1 = 1728 (p_0 + p_1 a'_1)^3 \left\{ e_1 + \frac{1}{1152} (\rho_3^2 - \rho_3) \frac{a'_1}{b} \right\},$$

the rational function e_1, and the radical a'_1 retaining their recent meanings: because no rational function F''_1 of four independent variables x_1, x_2, x_3, x_4, which cannot be reduced to the form thus assigned for a''_1, can have itself $2\alpha''_1$ values, α''_1 being a prime number greater than 1, if the number of values of the prime power $\mathrm{F}_1''^{\alpha''_1}$ be only 2. Nor can any other radical such as a''_2 of the same order enter into the expression of the irreducible function $b^{(m)}$; because this other radical would be obliged to be of one or other of the two forms following, namely either

$$a_2'' = (q_0 + q_1 a_1')(u_1 + \rho_3^2 u_2 + \rho_3 u_3),$$

or else

$$a_2'' = (q_0 + q_1 a_1')(u_1 + \rho_3 u_2 + \rho_3^2 u_3),$$

ρ_3 being the same cube-root of unity in these expressions, as in the expression for a_1''; and the product of the two last trinomial factors is symmetric,

$$(u_1 + \rho_3^2 u_2 + \rho_3 u_3)(u_1 + \rho_3 u_2 + \rho_3^2 u_3) = 144 e_2;$$

so that either the quotient $\dfrac{a_2''}{a_1''}$ or the product $a_2'' a_1''$ would be a two-valued function, which would be known when a_1' had been calculated, without any new extraction of radicals. At the same time, if we observe that

$$u_1 + u_2 + u_3 = a_2,$$

we see that the three values u_1, u_2, u_3 of the three-valued function $x_\alpha x_\beta + x_\gamma x_\delta$ can be expressed as rational functions of the radicals a_1'' and a_1', or as irrational functions of the second order of the coefficients a_1, a_2, a_3, a_4 of the proposed biquadratic equation, namely the following,

$$\begin{aligned}
u_1 &= \frac{1}{3}\left\{a_2 + \frac{a_1''}{p_0 + p_1 a_1'} + \frac{144 e_2 (p_0 + p_1 a')}{a_1''}\right\},\\
u_2 &= \frac{1}{3}\left\{a_2 + \frac{\rho_3 a_1''}{p_0 + p_1 a_1'} + \frac{144 e_2 (p_0 + p_1 a')}{\rho_3 a_1''}\right\},\\
u_3 &= \frac{1}{3}\left\{a_2 + \frac{\rho_3^2 a_1''}{p_0 + p_1 a_1'} + \frac{144 e_2 (p_0 + p_1 a')}{\rho_3^2 a_1''}\right\};
\end{aligned}$$

so that if the biquadratic equation can be resolved at all, by any finite combination of radicals and rational functions, the solution must begin by calculating a square-root a_1' and a cube-root a_1'', which are in all essential respects the same as those required for resolving that other equation of which u_1, u_2, u_3 are roots, namely the following cubic equation:

$$u^3 - a_2 u^2 + (a_1 a_3 - 4a_3)u + (4a_2 - a_1^2)a_4 - a_3^2 = 0;$$

which may also be thus written,

$$(u - \tfrac{1}{3}a_2)^2 - 48 e_2 (u - \tfrac{1}{3}a_2) - 128 e_1 = 0.$$

Reciprocally if u_1, u_2, u_3 be known, by the solution of this cubic equation, or in any other way, we can calculate a_1' and a_1'', without any new extraction of radicals, since if we put, for abridgment,

$$\begin{aligned}
t_1 &= u_2 - u_3 = (x_1 - x_2)(x_3 - x_4),\\
t_2 &= u_1 - u_3 = (x_1 - x_3)(x_2 - x_4),\\
t_3 &= u_1 - u_2 = (x_1 - x_4)(x_2 - x_3),
\end{aligned}$$

we have

$$a_1' = bt_1t_2t_3,$$

and

$$a_1'' = (p_0 + p_1bt_1t_2t_3)(u_1 + \rho_3^2u_2 + \rho_3u_3).$$

Again, it is important to observe, that if any one of the three quantities t_1, t_2, t_3, such as t_1, be given, the other two t_2, t_3, and also u_1, u_2, u_3, can be deduced from it, without any new extraction; because, in general, the difference of any two roots of a cubic equation is sufficient to determine rationally all the three roots of that equation: it must therefore be possible to express the radicals a_1' and a_1'' as rational functions of t_1; and accordingly we find

$$a_1' = bt_1(144e_2 - t_1^2),$$

and

$$a_1'' = \{p_0 + p_1bt_1(144e_2 - t_1^2)\}\left(\frac{\rho_3^2 - \rho_3}{2}t_1 + \frac{576e_1}{48e_2 - t_1^2}\right);$$

while t_1 may be reciprocally be expressed as follows,

$$t_1 = u_2 - u_3 = \tfrac{1}{3}(\rho_3 - \rho_3^2)\left\{\frac{a_1''}{p_0 + p_1a_1'} - \frac{144e_2(p_0 + p_1a_1')}{a_1''}\right\}.$$

Hence the most general irrational function of the second order,

$$f_1'' = b_0' + b_1'a_1'' + b_2'a_1''^2,$$

which can enter into the composition of $b^{(m)}$, and in which b_0', b_1', b_2' are functions of the first order, and of the forms

$$(b_0')_0 + (b_0')_1a_1', \quad (b_1')_0 + (b_1')_1a_1', \quad (b_2')_0 + (b_2')_1a_1',$$

may be considered as a rational function of t_1,

$$f_1'' = \phi(t_1) = \phi(\overline{x_1 - x_2} \,.\, \overline{x_3 - x_4});$$

it is, therefore, included under the form (II. II.), and is either six-valued or three-valued, according as it does not, or as it does reduce itself to a rational function of u_1, by becoming a rational function of t_1^2, and in neither case can it become a four-valued function such as x. We must therefore suppose, that the sought irrational expression $b^{(m)}$, for a root x of the general biquadratic, contains at least one radical a_1''' of the third order, which, relatively to the coefficients a_1, a_2, a_3, a_4, must be a square-root, (and not a cube-root,) of the form

$$a_1''' = \sqrt{b_0' + b_1'a_1'' + b_2'a_1''^2};$$

and, relatively to the roots x_1, x_2, x_3, x_4, must admit of being expressed either as a twelve-valued function, with a six-valued square, which square is of the form (II. II.); or else as a six-valued function, which is not itself of that form (II. II.), and of which the square is three-valued. This radical a_1''' must therefore admit of being put under the form

$$a_1''' = b_\alpha'' v_\alpha,$$

the factor b_α'' being a function of the second or of a lower order, and v_α being one or other of the three following functions,

$$v_1 = x_1 + x_2 - x_3 - x_4, \quad v_2 = x_1 + x_3 - x_2 - x_4, \quad v_3 = x_1 + x_4 - x_2 - x_3,$$

which are themselves six-valued, but have three-valued squares. And since the product of the three functions v_α is symmetric,

$$v_1 v_2 v_3 = 64 \,.\, e_4,$$

(e_4 having here the same meaning as in the second article,) we need only consider, at most, two radicals of the third order,

$$a_1''' = b_1'' v_1 = \sqrt{b_1''^2 (a_1^2 - 4a_2 + 4u_1)}, \quad a_2''' = b_2'' v_2 = \sqrt{b_2''^2 (a_1^2 - 4a_2 + 4u_2)};$$

and may express the most general irrational function of the third order, which can enter into the composition of $b^{(m)}$, as follows:

$$f_1''' = b_{0,0}'' + b_{1,0}'' a_1''' + b_{0,1}'' a_2''' + b_{1,1}'' a_1''' a_2''';$$

the coefficients of this expression being functions of the second or of lower orders, If we suppress entierely one of the two last radicals, such as a_2''', without introducing any higher radical a_1^{IV}, we shall indeed obtain a simplified expression, but cannot thereby represent any root, such as x_α, of the proposed biquadratic equation; for if we could do this, we should then have a system of two expressions for two different roots, x_α, x_β, of the forms

$$x_\alpha = b_0'' + b_1'' a_1''', \quad x_\beta = b_0'' - b_1'' a_1''',$$

which would give

$$b_0'' = \tfrac{1}{2}(x_\alpha + x_\beta);$$

but this last rational function, although six-valued, cannot be put under the form (II. II.), and therefore cannot be equal to any function of the second order, such as b_0''. Retaining therefore both the radicals, a_1''', a_2''', we have next to observe, that if the function f_1''' can coincide with the sought function $b^{(m)}$, so as to represent some one root x_α of the proposed biquadratic equation, it must give a system of expressions for all the four roots x_α, x_β, x_γ, x_δ, in some arrangement or other, by merely changing the signs of those two radicals of the third order; namely the following system,

$$\begin{aligned}
x_\alpha &= b_{0,0}'' + b_{1,0}'' a_1''' + b_{0,1}'' a_2''' + b_{1,1}'' a_1''' a_2''', \\
x_\beta &= b_{0,0}'' + b_{1,0}'' a_1''' - b_{0,1}'' a_2''' - b_{1,1}'' a_1''' a_2''', \\
x_\gamma &= b_{0,0}'' - b_{1,0}'' a_1''' + b_{0,1}'' a_2''' - b_{1,1}'' a_1''' a_2''', \\
x_\delta &= b_{0,0}'' - b_{1,0}'' a_1''' - b_{0,1}'' a_2''' + b_{1,1}'' a_1''' a_2''';
\end{aligned}$$

which four expressions for the four roots conduct to the four following relations,

$$
\begin{aligned}
b''_{0,0} &= \tfrac{1}{4}(x_\alpha + x_\beta + x_\gamma + x_\delta),\\
b''_{1,0}a'''_1 &= \tfrac{1}{4}(x_\alpha + x_\beta - x_\gamma - x_\delta),\\
b''_{0,1}a'''_2 &= \tfrac{1}{4}(x_\alpha - x_\beta + x_\gamma - x_\delta),\\
b''_{1,1}a'''_1a'''_2 &= \tfrac{1}{4}(x_\alpha - x_\beta - x_\gamma + x_\delta).
\end{aligned}
$$

Reciprocally, if these four last conditions can be satisfied, by any suitable arrangement of the four roots, and by any suitable choice of those coefficients or functions which have hitherto been left undetermined, we shall have the four expressions just now mentioned, for the four roots of the general biquadratic, as the four values of an irrational and irreducible function b''', of the third order, Now, these four conditions are satisfied when we suppose

$$x_\alpha = x_1, \quad x_\beta = x_2, \quad x_\gamma = x_3, \quad x_\delta = x_4;$$

$$b''_{0,0} = \frac{-a_1}{4}; \quad b''_{1,0} = \frac{1}{4b''_1}; \quad b''_{0,1} = \frac{1}{4b''_2};$$

and finally,

$$b''_{1,1} = \frac{16e_4}{b''_1b''_2v_1^2v_2^2};$$

but not by any suppositions essentially distinct from these. It is therefore possible to express the four roots of the general biquadratic equation, as the four values of an irrational and irreducible expression of the third order b''', namely as the following:

$$
\begin{aligned}
x_1 = b'''_{0,0} &= \frac{-a_1}{4} + \frac{a'''_1}{4b''_1} + \frac{a'''_2}{4b''_2} + \frac{16b''_1b''_2e_4}{a'''_1a'''_2};\\
x_2 = b'''_{0,1} &= \frac{-a_1}{4} + \frac{a'''_1}{4b''_1} - \frac{a'''_2}{4b''_2} - \frac{16b''_1b''_2e_4}{a'''_1a'''_2};\\
x_3 = b'''_{1,0} &= \frac{-a_1}{4} - \frac{a'''_1}{4b''_1} + \frac{a'''_2}{4b''_2} - \frac{16b''_1b''_2e_4}{a'''_1a'''_2};\\
x_4 = b'''_{1,1} &= \frac{-a_1}{4} - \frac{a'''_1}{4b''_1} - \frac{a'''_2}{4b''_2} + \frac{16b''_1b''_2e_4}{a'''_1a'''_2};
\end{aligned}
$$

and there exists no system of expressions, essentially distinct from these, which can express the same four roots, without the introduction of some radical, such as a_1^{IV}, of an order higher than the third. We must, however, remember that these expressions involve several arbitrary symmetric functions of x_1, x_2, x_3, x_4, or arbitrary rational functions of a_1, a_2, a_3, a_4, which enter into the composition of the radicals a'_1, a''_1, a'''_1, a'''_2, though only in the way of multiplying a function by an exact square or cube before the square-root or cube-root is extracted: namely, the quantity b in a'_1; p_0 and p_1 in a''_1; and, in the radicals a'''_1, a'''_2, twelve other arbitrary quantities, introduced by the functions b''_1, b''_2, which latter functions may be thus developed,

$$
\begin{aligned}
b''_1 &= r_{0,0} + r_{0,1}a'_1 + (r_{1,0} + r_{1,1}a'_1)a''_1 + (r_{2,0} + r_{2,1}a'_1)a''^2_1,\\
b''_2 &= r^{\backprime}_{0,0} + r^{\backprime}_{0,1}a'_1 + (r^{\backprime}_{1,0} + r^{\backprime}_{1,1}a'_1)a''_1 + (r^{\backprime}_{2,0} + r^{\backprime}_{2,1}a'_1)a''^2_1.
\end{aligned}
$$

In the earlier articles of this Essay, these fifteen arbitrary quantities had the following particular values,

$$b = \frac{\rho_3^2 - \rho_3}{1152}; \quad p_0 = \tfrac{1}{12}; \quad p_1 = 0;$$

$$r_{0,0} = \tfrac{1}{4}; \quad r_{0,1} = r_{1,0} = r_{1,1} = r_{2,0} = r_{2,1} = 0;$$

$$r^{\grave{}}_{0,0} = \tfrac{1}{4}; \quad r^{\grave{}}_{0,1} = r^{\grave{}}_{1,0} = r^{\grave{}}_{1,1} = r^{\grave{}}_{2,0} = r^{\grave{}}_{2,1} = 0.$$

Apparent differences between two systems of expressions of the third order, for the four roots of a biquadratic equation, may also arise from differences in the arrangement of those four roots.

Analogous reasonings, the details of which will easily suggest themselves to those who have studied the foregoing discussion, show that if we retain only one radical of the third order a_1''', but introduce a radical of the fourth order a_1^{IV}, for the purpose of obtaining the only other sort of irrational and irreducible expression $b^{(m)} = b^{\mathrm{IV}}$, which can represent a root of the same general biquadratic equation, we must then suppose this new radical a_1^{IV} to be a square-root, of the form

$$a_1^{\mathrm{IV}} = p'''(x_1 - x_2) = \sqrt{p'''^2 \left(-\frac{v_1^2}{4} + 12e_3 + \frac{32e_4}{v_1} \right)};$$

p''' being a function of the third or of a lower order, which in the earlier articles of this Essay had the particular value $\frac{1}{2}$; which v_1 has the meaning recently assigned, and e_3, e_4 have those which were stated in the second article; we must also employ the expressions

$$x_1 = b_0''' + b_1''' a_1^{\mathrm{IV}} = \frac{-a_1}{4} + \frac{v_1}{4} + \frac{a_1^{\mathrm{IV}}}{2p'''},$$
$$x_2 = b_0''' - b_1''' a_1^{\mathrm{IV}} = \frac{-a_1}{4} + \frac{v_1}{4} - \frac{a_1^{\mathrm{IV}}}{2p'''},$$

and

$$x_3 = \frac{-a_1}{4} - \frac{v_1}{4} + \frac{p''' t_1}{2a_1^{\mathrm{IV}}}, \quad x_4 = \frac{-a_1}{4} - \frac{v_1}{4} - \frac{p''' t_1}{2a_1^{\mathrm{IV}}},$$

t_1 retaining here its recent meaning; or, at least, we must make suppositions, and must employ expressions, not differing essentially from these.

But all the radicals a_1', a_1'', a_1''', a_2''', a_1^{IV}, introduced in the present article, agree in all essential respects with those which have been long employed, for the calculation of the roots of the general biquadratic equation; it is, therefore, impossible to discover any new expression for any one of those four roots, which, after being cleared from all superfluous extractions of radicals, shall differ essentially, in the extractions that remain, from the expressions that have been long discovered. And the only important difference, with respect to these extractions of radicals, between any two general methods for resolving biquadratic equations, if both be free from all superfluous extractions, is, that after calculating first, in both methods, a square-root a_1', and a cube-root a_1'', (operations which are equivalent to those required for the solution of an auxiliary cubic equation,) we may afterwards either calculate two *simultaneous* square-roots a_1''', a_2''', as in the method of EULER, or else two *successive* square-roots a_1''', a_1^{IV}, as in the method of FERRARI or DES CARTES: for, in the view in which they are here considered, the methods of these two last-mentioned mathematicians do not essentially differ from each other.

[21.] It is not necessary, for the purposes of the inquiry into the possibility or impossibility of representing, by any expression of the form $b^{(m)}$, a root x of the general equation of the fifth degree,

$$x^5 + a_1 x^4 + a_2 x^3 + a_3 x^2 + a_4 x + a_5 = 0,$$

to investigate all possible forms of rational functions of five variables, which have fewer than 120 values; but it is necessary to discover all those forms which have five or fewer values. Now, if the rational function

$$\mathrm{F}(x_1, x_2, x_3, x_4, x_5)$$

have fewer than six values, when the five arbitrary roots x_1, x_2, x_3, x_4, x_5, of the above-mentioned general equation are interchanged in all possible ways, it must, by a still stronger reason, have fewer than six values, when only the first four roots x_1, x_2, x_3, x_4, are interchanged in any manner, the fifth root x_5 remaining unchanged.

Hence, by the properties of functions of four variables, the function F must be reducible to one of the four following forms, corresponding to those which, in the nineteenth article, were marked (I. III), (II. III), (I. II.), and (I. I.):

(a) $\phi(x_5)$;

(b) $\phi(x_5, \overline{x_1 - x_2} \,.\, \overline{x_1 - x_3} \,.\, \overline{x_1 - x_4} \,.\, \overline{x_2 - x_3} \,.\, \overline{x_2 - x_4} \,.\, \overline{x_3 - x_4})$;

(c) $\phi(x_5, x_1 x_2 + x_3 x_4)$;

(d) $\phi(x_5, x_4)$;

or at least to some form not essentially distinct from these. In making this reduction, the following principle is employed, that any symmetric function of x_1, x_2, x_3, x_4, is a rational function of x_5 and of the five coefficients a_1, a_2, a_3, a_4, a_5; which latter coefficients are tacitly supposed to be capable of entering in any manner into the rational functions ϕ.

It may also be useful to remark, before going farther, that the four forms here referred to, of functions of four variables, with four or fewer values, may be deduced anew as follows. Retaining the abridged notation $(\alpha, \beta, \gamma, \delta)$, we see immediately that if the six syntypical functions

$$(1,2,3,4), \quad (2,3,1,4), \quad (3,1,2,4), \quad (1,3,2,4), \quad (3,2,1,4), \quad (2,1,3,4)$$

be not unequal among themselves, they must either all be equal, in which case we have the four-valued form $\phi(x_4)$ or (I. I.), or else must distribute themselves into two distinct groups of three, or into three distinct groups of two equal functions. But if we suppose

$$(1,2,3,4) = (2,3,1,4) = (3,1,2,4),$$

in order to get the reduction to two groups, the functions $(1, 2, 3, 4)$ and $(2, 1, 3, 4)$ being not yet supposed to be equal; and then require that the six following values of $(\alpha, \beta, \gamma, \delta)$,

$$(1, 2, 3, 4), \quad (2, 1, 3, 4), \quad (1, 2, 4, 3), \quad (2, 1, 4, 3), \quad (1, 3, 4, 2), \quad (3, 1, 4, 2),$$

shall not be all unequal; we must either make some supposition, such as

$$(1, 2, 3, 4) = (1, 2, 4, 3),$$

which conducts to the one-valued form (I. III.), or else must make some supposition, such as

$$(1, 2, 3, 4) = (2, 1, 4, 3),$$

which conducts to the two-valued form (II. III.). And if we suppose

$$(1, 2, 3, 4) = (2, 1, 3, 4),$$

in order to reduce the six functions $(1, 2, 3, 4), \ldots (2, 1, 3, 4)$ to three distinct groups, the functions $(1, 2, 3, 4)$ and $(2, 3, 1, 4)$ being supposed unequal; and then require that of the six following values,

$$(1, 2, 3, 4), \quad (2, 3, 1, 4), \quad (3, 1, 2, 4), \quad (1, 2, 4, 3), \quad (2, 4, 1, 3), \quad (4, 1, 2, 3),$$

there shall be fewer than five unequal; we must either suppose

$$(2, 3, 1, 4) = (4, 1, 2, 3),$$

in which case we are conducted to the three-valued form (I. II.); or else must suppose

$$(2, 3, 1, 4) = (2, 4, 1, 3),$$

which conducts again to the four-valued function (I. I.), by giving $(1, 2, 3, 4) = \phi(x_3)$.

Now of the four forms (a), (b), (c), (d), the form (a) is five-valued, and therefore admissible in the present inquiry; but the form (b) is, in general, ten-valued; the form (c) has, in general fifteen values; and the form (d) has twenty. If, then, we are to reduce the functions (b) (c) (d) within that limit of number of values to which we are at present confining ourselves, we must restrict them by some new conditions, of which the following are sufficient types:

$$
\begin{aligned}
&(\mathrm{b})' && \phi(x_5, \overline{x_1 - x_2} \,.\, \overline{x_1 - x_3} \,.\, \overline{x_1 - x_4} \,.\, \overline{x_2 - x_3} \,.\, \overline{x_2 - x_4} \,.\, \overline{x_3 - x_4}) \\
& && \quad = \phi(x_5, -\overline{x_1 - x_2} \,.\, \overline{x_1 - x_3} \,.\, \overline{x_1 - x_4} \,.\, \overline{x_2 - x_3} \,.\, \overline{x_2 - x_4} \,.\, \overline{x_3 - x_4}); \\
&(\mathrm{b})'' && \phi(x_5, \overline{x_1 - x_2} \,.\, \overline{x_1 - x_3} \,.\, \overline{x_1 - x_4} \,.\, \overline{x_2 - x_3} \,.\, \overline{x_2 - x_4} \,.\, \overline{x_3 - x_4}) \\
& && \quad = \phi(x_4, \overline{x_1 - x_2} \,.\, \overline{x_1 - x_3} \,.\, \overline{x_1 - x_5} \,.\, \overline{x_2 - x_3} \,.\, \overline{x_2 - x_5} \,.\, \overline{x_3 - x_5}); \\
&(\mathrm{b})''' && \phi(x_5, \overline{x_1 - x_2} \,.\, \overline{x_1 - x_3} \,.\, \overline{x_1 - x_4} \,.\, \overline{x_2 - x_3} \,.\, \overline{x_2 - x_4} \,.\, \overline{x_3 - x_4}) \\
& && \quad = \phi(x_4, -\overline{x_1 - x_2} \,.\, \overline{x_1 - x_3} \,.\, \overline{x_1 - x_5} \,.\, \overline{x_2 - x_3} \,.\, \overline{x_2 - x_5} \,.\, \overline{x_3 - x_5}); \\
&(\mathrm{c})' && \phi(x_5, x_1x_2 + x_3x_4) = \phi(x_5, x_1x_3 + x_2x_4); \\
&(\mathrm{c})'' && \phi(x_5, x_1x_2 + x_3x_4) = \phi(x_4, x_1x_2 + x_3x_5); \\
&(\mathrm{c})''' && \phi(x_5, x_1x_2 + x_3x_4) = \phi(x_4, x_1x_3 + x_2x_5); \\
&(\mathrm{d})' && \phi(x_5, x_4) = \phi(x_5, x_3); \\
&(\mathrm{d})'' && \phi(x_5, x_4) = \phi(x_4, x_3); \\
&(\mathrm{d})''' && \phi(x_5, x_4) = \phi(x_2, x_3).
\end{aligned}
$$

(To suppose $\phi(x_5, x_4) = \phi(x_4, x_5)$, would indeed reduce the number of values of the function (d) from twenty to ten, but a new reduction would be required, in order to depress that number below six, and thus we should still be obliged to employ one of the three conditions (d)$'$ (d)$''$ (d)$'''$.) Of these twelve different conditions (b)$'$ ... (d)$'''$, some one of which we must employ, (or at least some condition not essentially different from it,) the three marked (b)$'$ (c)$'$ (d)$'$ are easily seen to reduce respectively the three functions (b) (c) (d) to the five-valued form (a); they are therefore admissible, but they give no new information. The supposition (b)$''$ conducts us to equate the function (b) to the following,

$$\phi(x_3, \overline{x_1 - x_2} \,.\, \overline{x_1 - x_5} \,.\, \overline{x_1 - x_4} \,.\, \overline{x_2 - x_5} \,.\, \overline{x_2 - x_4} \,.\, \overline{x_5 - x_4}),$$

because it allows us to interchange x_5 and x_3, inasmuch as x_3 may previously be put in the place of x_4 and x_4 in the place of x_3, by interchanging at the same time x_1 and x_2,—a double interchange which does not alter the product $\overline{x_1 - x_3} \ldots \overline{x_3 - x_4}$, since it only changes simultaneously the signs of the two factors $x_1 - x_2$ and $x_3 - x_4$; or because, if we denote the function (b) by the symbol $(1, 2, 3, 4, 5)$, we have

$$(1, 2, 3, 4, 5) = (2, 1, 4, 3, 5),$$

and also by (b)$''$,

$$(1, 2, 3, 4, 5) = (1, 2, 3, 5, 4),$$

so that we must have

$$(1, 2, 3, 4, 5) = (2, 1, 4, 5, 3) = (1, 2, 5, 4, 3);$$

but also the condition (b)$''$ gives

$$(1, 2, 5, 4, 3) = (1, 2, 5, 3, 4);$$

we must therefore suppose

$$(1, 2, 3, 5, 4) = (1, 2, 5, 3, 4),$$

that is

$$\begin{aligned}&\phi(x_4, \overline{x_1 - x_2} \,.\, \overline{x_1 - x_3} \,.\, \overline{x_1 - x_5} \,.\, \overline{x_2 - x_3} \,.\, \overline{x_2 - x_5} \,.\, \overline{x_3 - x_5}) \\ &\quad = \phi(x_4, -\overline{x_1 - x_2} \,.\, \overline{x_1 - x_3} \,.\, \overline{x_1 - x_5} \,.\, \overline{x_2 - x_3} \,.\, \overline{x_2 - x_5} \,.\, \overline{x_3 - x_5}),\end{aligned}$$

which is an equation of the form (b)$'$, and reduces the function (b) to the form (a), and ultimately to a symmetric function a, because x_5 and x_4 may be interchanged. The supposition (b)$'''$ conducts to a two-valued function, which changes value when any two of the five roots are interchanged, so that the sum $(1, 2, 3, 4, 5) + (1, 2, 3, 5, 4)$, and the quotient

$$\frac{(1, 2, 3, 4, 5) - (1, 2, 3, 5, 4)}{(x_1 - x_2)(x_1 - x_3) \ldots (x_4 - x_5)},$$

are some symmetric functions, which may be called $2a$ and $2b$; we have therefore, in this case, a function of the form,

(e) $\quad a + b\,\overline{x_1 - x_2} . \overline{x_1 - x_3} . \overline{x_1 - x_4} . \overline{x_1 - x_5} . \overline{x_2 - x_3} . \overline{x_2 - x_4} . \overline{x_2 - x_5} . \overline{x_3 - x_4} . \overline{x_3 - x_5} . \overline{x_4 - x_5},$

in which a and b are symmetric. The remaining suppositions, (c)$''$, (c)$'''$, (d)$''$, (d)$'''$, are easily seen to conduct only to symmetric functions; for instance, (c)$''$ gives

$$\begin{aligned}\phi(x_5, x_1x_2 + x_3x_4) &= \phi(x_4, x_3x_5 + x_2x_1) = \phi(x_1, x_3x_5 + x_2x_4)\\ &= \phi(x_1, x_2x_4 + x_3x_5) = \phi(x_5, x_2x_4 + x_3x_1) = \phi(x_5, x_1x_3 + x_2x_4),\end{aligned}$$

so that the condition (c)$'$ is satisfied, and at the same time x_5 is interchangeable with x_4. And it is easy to see that the five-valued function $\phi(x_\alpha)$ may be put under the form

$$\text{(f)}\quad b_0 + b_1x_\alpha + b_2x_\alpha^2 + b_3x_\alpha^3 + b_4x_\alpha^4;$$

the coefficients b_0, b_1, b_2, b_3, b_4 being symmetric. It is clear also that neither this five-valued function (f), nor the two-valued function (e), admits of any reduction in respect to number of values, without becoming altogether symmetric. There are, therefore, no unsymmetric and rational functions of five independent variables, with fewer than six values, except only the two-valued function (e), and the five-valued function (f).

Suppose now that we have the equation

$$a'_1 = \mathrm{F}'_1(x_1, x_2, x_3, x_4, x_5),$$

F'_1 being a rational but unsymmetric function; and that

$$a_1'^{\alpha'_1} = f_1(a_1, a_2, a_3, a_4, a_5),$$

the exponent α'_1 being prime, and the function f_1 being rational relatively to $a_1, \dots a_5$, and therefore symmetric relatively to $x_1, \dots x_5$. With these suppositions, the function F'_1 must, by the principles of a former article, have exactly α'_1 values, corresponding to changes of arrangement of the five arbitrary quantities $x_1, \dots x_5$; the exponent α'_1 must therefore be a prime divisor of the product 120 ($= 1.2.3.4.5$); that is, it must be 2, or 3, or 5. But we have seen that no rational function of five variables has exactly three values; and if we supposed it to have five values, so as to put, (by what has been already shewn,)

$$a'_1 = b_0 + b_1x_\alpha + b_2x_\alpha^2 + b_3x_\alpha^3 + b_4x_\alpha^4,$$

we should then have three other equations of the forms

$$\begin{aligned}a_1'^2 &= b_0^{(2)} + b_1^{(2)}x_\alpha + b_2^{(2)}x_\alpha^2 + b_3^{(2)}x_\alpha^3 + b_4^{(2)}x_\alpha^4,\\ a_1'^3 &= b_0^{(3)} + b_1^{(3)}x_\alpha + b_2^{(3)}x_\alpha^2 + b_3^{(3)}x_\alpha^3 + b_4^{(3)}x_\alpha^4,\\ a_1'^4 &= b_0^{(4)} + b_1^{(4)}x_\alpha + b_2^{(4)}x_\alpha^2 + b_3^{(4)}x_\alpha^3 + b_4^{(4)}x_\alpha^4,\end{aligned}$$

the coefficients being all symmetric, and being determined through the elimination of all higher powers of x_α than the fourth, by means of the equations

$$\begin{gathered}x_\alpha^5 + a_1x_\alpha^4 + a_2x_\alpha^3 + a_3x_\alpha^2 + a_4x_\alpha + a_5 = 0,\\ x_\alpha^6 + a_1x_\alpha^5 + a_2x_\alpha^4 + a_3x_\alpha^3 + a_4x_\alpha^2 + a_5x_\alpha = 0, \quad \text{\&c.};\end{gathered}$$

and it would always be possible to find symmetric multipliers c_1, c_2, c_3, c_4, which would not all be equal to 0, and would be such that

$$\begin{aligned}
c_1 b_2 + c_2 b_2^{(2)} + c_3 b_2^{(3)} + c_4 b_2^{(4)} &= 0,\\
c_1 b_3 + c_2 b_3^{(2)} + c_3 b_3^{(3)} + c_4 b_3^{(4)} &= 0,\\
c_1 b_4 + c_2 b_4^{(2)} + c_3 b_4^{(3)} + c_4 b_4^{(4)} &= 0;
\end{aligned}$$

in this manner then we should obtain an equation of the form

$$c_1 a_1' + c_2 a_1'^2 + c_3 a_1'^3 + c_4 a_1'^4 = c_1 b_0 + c_2 b_0^{(2)} + c_3 b_0^{(3)} + c_4 b_0^{(4)} + (c_1 b_1 + c_2 b_1^{(2)} + c_3 b_1^{(3)} + c_4 b_1^{(4)}) x_\alpha,$$

in which it is impossible that the coefficient of x_α should vanish, because the five unequal values of a_1' could not all satisfy one common equation, of the fourth or of a lower degree; we should therefore have an expression for x_α of the form

$$x_\alpha = d_0 + d_1 a_1' + d_2 a_1'^2 + d_3 a_1'^3 + d_4 a_1'^4,$$

the coefficients $d_0, \dots d_4$ being symmetric; and for the same reason we should have also

$$\begin{aligned}
x_\beta &= d_0 + d_1 \rho_5 a_1' + d_2 \rho_5^2 a_1'^2 + d_3 \rho_5^3 a_1'^3 + d_4 \rho_5^4 a_1'^4,\\
x_\gamma &= d_0 + d_1 \rho_5^2 a_1' + d_2 \rho_5^4 a_1'^2 + d_3 \rho_5 a_1'^3 + d_4 \rho_5^3 a_1'^4,\\
x_\delta &= d_0 + d_1 \rho_5^3 a_1' + d_2 \rho_5 a_1'^2 + d_3 \rho_5^4 a_1'^3 + d_4 \rho_5^2 a_1'^4,\\
x_\epsilon &= d_0 + d_1 \rho_5^4 a_1' + d_2 \rho_5^3 a_1'^2 + d_3 \rho_5^2 a_1'^3 + d_4 \rho_5 a_1'^4,
\end{aligned}$$

x_α, x_β, x_γ, x_δ, x_ϵ denoting, in some arrangement or other, the five roots x_1, x_2, x_3, x_4, x_5, and ρ_5, ρ_5^2, ρ_5^3, ρ_5^4 being the four imaginary fifth-roots of unity; consequently we should have

$$5 d_1 a_1' = x_\alpha + \rho_5^4 x_\beta + \rho_5^3 x_\gamma + \rho_5^2 x_\delta + \rho_5 x_\epsilon;$$

a result which is absurd, the second member of the equation having 120 values, while the first member has only five. We must therefore suppose that the exponent α_1' is $= 2$, and consequently must adopt the expression

$$a_1' = b(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_1 - x_5)(x_2 - x_3)(x_2 - x_4)(x_2 - x_5)(x_3 - x_4)(x_3 - x_5)(x_4 - x_5),$$

the factor b being symmetric. This, therefore, is the only rational and unsymmetric function of five arbitrary quantities, which has a prime power (namely its square) symmetric.

Let us next inquire whether it be possible to find any unsymmetric but rational function,

$$a_1'' = \mathrm{F}_1''(x_1, x_2, x_3, x_4, x_5),$$

which, having itself more than two values, shall have a prime power two-valued,

$$a_1''^{\alpha_1''} = f_1' = a + b(x_1 - x_2) \dots (x_4 - x_5).$$

If so, the function F''_1 must have exactly $2\alpha''_1$ values, and consequently the prime exponent α''_1 must be either three or five, because it must be a divisor of 120, and cannot be $= 2$, since no rational function of five arbitrary quantities has exactly four values: so that a''_1 or F''_1 must be either a cube-root of a fifth-root of the two-valued function f'_1. And the six or ten values of F''_1 must admit of being expressed as follows:

$$(1,2,3,4,5)_i; \quad \rho_{\alpha''_1}(1,2,3,4,5)_i; \ \ldots \ \rho_{\alpha''_1}^{\alpha''_1-1}(1,2,3,4,5)_i;$$

$$(1,2,3,4,5)_k; \quad \rho^{\backprime}_{\alpha''_1}(1,2,3,4,5)_k; \ \ldots \ \rho_{\alpha''_1}^{\backprime\alpha''_1-1}(1,2,3,4,5)_k;$$

in which $\rho_{\alpha''_1}$ and $\rho^{\backprime}_{\alpha''_1}$ are imaginary cube-roots or fifth-roots of unity, according as α''_1 is 3 or 5; while $(1,2,3,4,5)_i$ and $(1,2,3,4,5)_k$ are some two different values of the function F''_1, which may be called F''_1 and $\mathrm{F}^{\backprime\prime\prime}_1$, and correspond to different arrangements of x_1, x_2, x_3, x_4, x_5, being also such that

$$\begin{aligned}
\mathrm{F}_1^{\prime\prime\alpha''_1} &= (1,2,3,4,5)_i^{\alpha''_1} = a + b(x_1 - x_2)\ldots(x_4 - x_5),\\
\mathrm{F}_1^{\backprime\prime\prime\alpha''_1} &= (1,2,3,4,5)_k^{\alpha''_1} = a - b(x_1 - x_2)\ldots(x_4 - x_5).
\end{aligned}$$

These last equations show that the cube or fifth power (according as a''_1 is 3 or 5) of the product of $(1,2,3,4,5)_i$ and $(1,2,3,4,5)_k$ is symmetric, and consequently, by what was lately proved, that this product itself is symmetric; so that we may write

$$\mathrm{F}''_1 \,.\, \mathrm{F}^{\backprime\prime\prime}_1 = (1,2,3,4,5)_i \,.\, (1,2,3,4,5)_k = c,$$

and therefore

$$\nabla(1,2,3,4,5)_i \,.\, \nabla(1,2,3,4,5)_k = c,$$

∇ being here the characteristic of any arbitrary change of arrangement of the five roots, which change, however, is to operate similarly on the two functions to which the symbol is prefixed. (For example, if we suppose

$$(1,2,3,4,5)_i = (1,2,3,5,4), \quad (1,2,3,4,5)_k = (1,2,4,3,5),$$

and if we employ ∇ to indicate that change which consists in altering the first to the second, the second to the third, the third to the fourth, the fourth to the fifth, and the fifth to the first of the five roots in any one arrangement, we shall have, in the present notation,

$$\nabla(1,2,3,4,5)_i = (2,3,5,4,1), \quad \nabla(1,2,3,4,5)_k = (2,4,3,5,1);$$

and similarly in other cases.) Supposing then that ∇ denotes the change of arrangement of the five roots which is made in passing from that value of the function F''_1 which is $= (1,2,3,4,5)_i$ to that other value of the same function which is $= \rho_{\alpha''_1}(1,2,3,4,5)_i$, we see that the same change performed on $(1,2,3,4,5)_k$ must multiply this latter value not by ρ''_α but by $\rho^{-1}_{\alpha''_1}$;

which factor is, however, of the form $\rho^{\backslash}_{\alpha_1''}$, so that we may denote the $2\alpha_1''$ values of F_1'' as follows:

$$(1,2,3,4,5)_i; \quad \nabla(1,2,3,4,5)_i; \quad \dots \quad \nabla^{\alpha_1''-1}(1,2,3,4,5)_i;$$

$$(1,2,3,4,5)_k; \quad \nabla(1,2,3,4,5)_k; \quad \dots \quad \nabla^{\alpha_1''-1}(1,2,3,4,5)_k.$$

We see, at the same time, that the sum of the two functions $(1,2,3,4,5)_i$ and $(1,2,3,4,5)_k$ admits of *at least* α_1'' different values, namely,

$$\nabla^0\{(1,2,3,4,5)_i + (1,2,3,4,5)_k\} = \mathrm{F}_1'' + \mathrm{F}_1^{\backslash\prime\prime},$$

$$\nabla^1\{(1,2,3,4,5)_i + (1,2,3,4,5)_k\} = \rho_{\alpha_1''}\mathrm{F}_1'' + \rho_{\alpha_1''}^{-1}\mathrm{F}_1^{\backslash\prime\prime},$$

$$\dots\dots\dots$$

$$\nabla^{\alpha_1''-1}\{(1,2,3,4,5)_i + (1,2,3,4,5)_k\} = \rho_{\alpha_1''}^{\alpha_1''-1}\mathrm{F}_1'' + \rho_{\alpha_1''}^{-(\alpha_1''-1)}\mathrm{F}_1^{\backslash\prime\prime},$$

On the other hand, this sum $\mathrm{F}_1'' + \mathrm{F}_1^{\backslash\prime\prime}$ cannot admit of *more* than α_1'' values, because it must satisfy an equation of the degree α_1'', with symmetric coefficients; which results from the two relations

$$\mathrm{F}_1^{\prime\prime\alpha_1''} + \mathrm{F}_1^{\backslash\prime\prime\alpha_1''} = 2a, \quad \mathrm{F}_1''\mathrm{F}_1^{\backslash\prime\prime} = c,$$

and is either the cubic equation

$$(\mathrm{F}_1'' + \mathrm{F}_1^{\backslash\prime\prime})^3 - 3c(\mathrm{F}_1'' + \mathrm{F}_1^{\backslash\prime\prime}) - 2a = 0,$$

or the equation of the fifth degree

$$(\mathrm{F}_1'' + \mathrm{F}_1^{\backslash\prime\prime})^5 - 5c(\mathrm{F}_1'' + \mathrm{F}_1^{\backslash\prime\prime})^3 + 5c^2(\mathrm{F}_1'' + \mathrm{F}_1^{\backslash\prime\prime}) - 2a = 0,$$

according as α_1'' is 3 or 5. We must therefore suppose that the function $\mathrm{F}_1'' + \mathrm{F}_{\backslash\prime\prime}$ has *exactly* α_1'' values, and consequently that α_1'' is 5 and not 3, because no rational function of five independent variables has exactly three values. And from the form and properties of the only five-valued function of five variables, we must suppose farther, that

$$\mathrm{F}_1'' + \mathrm{F}_1^{\backslash\prime\prime} = \mathrm{F}_1'' + \frac{c}{\mathrm{F}_1''} = b_0 + b_1 x_\alpha + b_2 x_\alpha^2 + b_3 x_\alpha^3 + b_4 x_\alpha^4,$$

x_α being some one of the five roots $x_1, \dots x_5$, and the coefficients $b_0, \dots b_4$ being symmetric; and that conversely the root x_α may be thus expressed,

$$x_\alpha = d_0 + d_1\left(\mathrm{F}_1'' + \frac{c}{\mathrm{F}_1''}\right) + d_2\left(\mathrm{F}_1'' + \frac{c}{\mathrm{F}_1''}\right)^2 + \dots + d_4\left(\mathrm{F}_1'' + \frac{c}{\mathrm{F}_1''}\right)^4,$$

the coefficients $d_0, \dots d_4$ being symmetric. We must also suppose that by changing F_1'', successively, to $\rho_5\mathrm{F}_1''$, $\rho_5^2\mathrm{F}_1''$, $\rho_5^3\mathrm{F}_1''$, $\rho_5^4\mathrm{F}_1''$, we shall obtain successively, expressions for the other four roots x_β, x_γ, x_δ, x_ϵ, in some arrangement or other; and therefore, if we observe that $\mathrm{F}_1^{\prime\prime 5}$ has been concluded to be a function of the two-valued form, we find ourselves obliged to suppose that the five roots may be expressed as follows, (if the supposition under inquiry be correct,)

$$
\begin{aligned}
x_\alpha &= e'_0 + e'_1 \mathrm{F}''_1 + e'_2 \mathrm{F}''^2_1 + e'_3 \mathrm{F}''^3_1 + e'_4 \mathrm{F}''^4_1,\\
x_\beta &= e'_0 + \rho_5 e'_1 \mathrm{F}''_1 + \rho_5^2 e'_2 \mathrm{F}''^2_1 + \rho_5^3 e'_3 \mathrm{F}''^3_1 + \rho_5^4 e'_4 \mathrm{F}''^4_1,\\
x_\gamma &= e'_0 + \rho_5^2 e'_1 \mathrm{F}''_1 + \rho_5^4 e'_2 \mathrm{F}''^2_1 + \rho_5 e'_3 \mathrm{F}''^3_1 + \rho_5^3 e'_4 \mathrm{F}''^4_1,\\
x_\delta &= e'_0 + \rho_5^3 e'_1 \mathrm{F}''_1 + \rho_5 e'_2 \mathrm{F}''^2_1 + \rho_5^4 e'_3 \mathrm{F}''^3_1 + \rho_5^2 e'_4 \mathrm{F}''^4_1,\\
x_\epsilon &= e'_0 + \rho_5^4 e'_1 \mathrm{F}''_1 + \rho_5^3 e'_2 \mathrm{F}''^2_1 + \rho_5^2 e'_3 \mathrm{F}''^3_1 + \rho_5 e'_4 \mathrm{F}''^4_1,
\end{aligned}
$$

$e'_0, \dots e'_5$ being either symmetric or two-valued; but these expressions conduct to the absurd result,

$$5 e'_1 \mathrm{F}''_1 = x_\alpha + \rho_5^4 x_\beta + \rho_5^3 x_\gamma + \rho_5^2 x_\delta + \rho_5 x_\epsilon,$$

in which the first member has only ten, while the second member has 120 values. We are therefore obliged to reject as inadmissible the supposition

$$\mathrm{F}_1''^{\alpha''_1} = f'_1;$$

and we find that no rational function of five arbitrary variables can have any prime power two-valued, if its own values be more numerous than two.

[22.] There is now no difficulty in proving, after the manner of ABEL, that it is impossible to represent a root of the general equation of the fifth degree, as a function of the coefficients of that equation, by any expression of the form $b^{(m)}$; that is, by any finite combination of radicals and rational functions.

For, in the first place, since the coefficients $a_1, \dots a_5$ are symmetric functions of the roots $x_1, \dots x_5$, it is clear that we cannot express any one of the latter as a rational function of the former; m in $b^{(m)}$, must therefore be greater than 0; and the expression $b^{(m)}$ if it exist at all, must involve at least one radical of the first order, a'_1, which must admit of being expressed as a rational but unsymmetric function F'_1 of the five roots, but must have a prime power $\mathrm{F}_1'^{\alpha'_1}$ symmetric, and consequently must be a square-root, of the form deduced in the last article, namely,

$$a'_1 = b(x_1 - x_2) \dots (x_4 - x_5),$$

the factor b being symmetric. And because any other radical of the same order, a'_2, might be deduced from a'_1 by a multiplication such as the following, $a'_2 = \frac{c}{b} a'_1$, we see that no such other radical a'_2, of the first order, can enter into the expression $b^{(m)}$, when that expression is cleared of all superfluous functional radicals. On the other hand, a two-valued expression such as

$$f'_1 = b_0 + b_1 a'_1$$

cannot represent the five-valued function x; if then the sought expression $x = b^{(m)}$ exist at all, it must involve some radical of the second order, a''_1, and this radical must admit of being expressed as a rational function of the five roots, which function is to have, itself, more than two values, but to have some prime power $\mathrm{F}_1''^{\alpha''_1}$, two-valued. And since it has been proved that no such function F_1'' exists, it follows that no function of the form $b^{(m)}$ can represent the sought root x of the general equation of the fifth degree. If then that general equation admit of being resolved at all, it must be by some process distinct from any finite combination of the operations of adding, subtracting, multiplying, dividing, elevating to powers, and extracting roots of functions.

[23.] It is, therefore impossible to satisfy the equation

$$b^{(m)5} + a_1 b^{(m)4} + a_2 b^{(m)3} + a_3 b^{(m)2} + a_4 b^{(m)} + a_5 = 0,$$

by any finite irrational function $b^{(m)}$; the five coefficients a_1, a_2, a_3, a_4, a_5 being supposed to remain arbitrary and independent. And, by still stronger reason, it is impossible to satisfy the equation

$$b^{(m)n} + a_1 b^{(m)n-1} + \cdots + a_{n-1} b^{(m)} + a_n = 0,$$

if n be greater than five, and $a_1, \ldots a_n$ arbitrary. For if we could do this, then the irrational function $b^{(m)}$ would, by the principles already established, have exactly n values; of which $n-5$ values would vanish when we supposed $a_n, a_{n-1}, \ldots a_6$ to become $= 0$, and the remaining five values would represent the five roots of the general equation of the fifth degree; but such a representation of the roots of that equation has been already proved to be impossible.

[24.] Although the whole of the foregoing has been suggested by that of Abel, and may be said to be a commentary thereon; yet it will not fail to be perceived, that there are several considerable differences between the one method of proof and the other. More particularly, in establishing the cardinal proposition that every radical in every irreducible expression for any one of the roots of any general equation is a rational function of those roots, it has appeared to the writer of this paper more satisfactory to begin by showing that the radicals of highest order will have that property, if those of lower orders have it, descending thus to radicals of the lowest order, and afterwards ascending again; than to attempt, as Abel has done, to prove the theorem, in the first instance, for radicals of the highest order. In fact, while following this last-mentioned method, Abel has been led to assume that the coefficient of the first power of some highest radical can always be rendered equal to unity, by introducing (generally) a new radical, which in the notation of the present paper may be expressed as follows:

$$\sqrt[\alpha_k^{(m)}]{\left\{ \sum_{\substack{\beta_i^{(m)} < \alpha_i^{(m)} \\ \beta_k^{(m)} = 1}} .\left(b^{(m-1)}_{\beta_1^{(m)}, \ldots \beta_{n^{(m)}}^{(m)}} \cdot a_1^{(m)\beta_1^{(m)}} \ldots a_{n^{(m)}}^{(m)\beta_{n^{(m)}}^{(m)}} \right) \right\}^{\alpha_k^{(m)}}} ;$$

but although the quantity under the radical sign, in this expression is indeed free from that irrationality of the m^{th} order which was introduced by the radical $a_k^{(m)}$, it is not, in general, free from the irrationalities of the same order introduced by the other radicals $a_1^{(m)}, \ldots$ of that order; and consquently the new radical, to which this process conducts, is in general elevated to the order $m+1$; a circumstance which Abel does not appear to have remarked, and which renders it difficult to judge of the validity of his subsequent reasoning. And because the other chief obscurity in ABEL's argument (in the opinion of the present writer) is connected with the proof of the theorem, that a rational function of five independent variables cannot have five values, and five only, unless it be symmetric relatively to four of its five elements; it has been thought advantageous, in the present paper, as preliminary to the discussion of the forms of functions of five arbitrary quantities, to establish certain auxiliary theorems respecting functions of fewer variables; which have served also to determine *à priori* all possible solutions (by radicals and rational functions) of all general algebraic equations below the fifth degree.

[25.] However, it may be proper to state briefly here the simple and elegant reasoning by which ABEL, after CAUCHY, has proved that if a function of five variables have fewer than five values, it must be either two-valued or symmetric. Let the function be for brevity denoted by $(\alpha, \beta, \gamma, \delta, \epsilon)$; and let ∇ and $\nabla^{\backprime}$ denote such changes, that

$$(\beta, \gamma, \delta, \epsilon, \alpha) = \nabla(\alpha, \beta, \gamma, \delta, \epsilon),$$
$$(\beta, \epsilon, \alpha, \gamma, \delta) = \nabla^{\backprime}(\alpha, \beta, \gamma, \delta, \epsilon).$$

These two changes are such that we have the two symbolic equations

$$\nabla^5 = 1, \quad \nabla^{\backprime 5} = 1;$$

but also, by supposition, some two of the five functions

$$\nabla^0(\alpha, \beta, \gamma, \delta, \epsilon), \quad \dots \quad \nabla^4(\alpha, \beta, \gamma, \delta, \epsilon)$$

are equal among themselves, and so are some two of the five functions

$$\nabla^{\backprime 0}(\alpha, \beta, \gamma, \delta, \epsilon), \quad \dots \quad \nabla^{\backprime 4}(\alpha, \beta, \gamma, \delta, \epsilon);$$

we have therefore two equations of the forms

$$\nabla^r = 1, \quad \nabla^{\backprime r^{\backprime}} = 1,$$

in which r and $r^{\backprime}$ are each greater than 0, but less than 5; and by combining these equations with the others just now found, we obtain

$$\nabla = 1, \quad \nabla^{\backprime} = 1:$$

that is

$$(\beta, \gamma, \delta, \epsilon, \alpha) = (\alpha, \beta, \gamma, \delta, \epsilon), \quad \text{and} \quad (\beta, \epsilon, \alpha, \gamma, \delta) = (\alpha, \beta, \gamma, \delta, \epsilon).$$

Hence

$$(\gamma, \alpha, \beta, \delta, \epsilon) = (\beta, \gamma, \delta, \epsilon, \alpha) = (\alpha, \beta, \gamma, \delta, \epsilon);$$

and in like manner,

$$(\alpha, \gamma, \delta, \beta, \epsilon) = (\alpha, \beta, \gamma, \delta, \epsilon) = (\gamma, \alpha, \beta, \delta, \epsilon);$$

we may therefore interchange the first and second of the five elements of the function, if we at the same time interchange either the second and third, or the third and fourth; and a similar reasoning shows that we may interchange any two, if we at the same time interchange any two others. An even number of such interchanges leaves therefore the function unaltered; but every alteration of arrangement of the five elements may be made by either an odd or an even number of such interchanges: the function, therefore, is either two-valued or symmetric; it having been supposed to have fewer than five values. Indeed, this is only a particular case of a more general theorem of CAUCHY, which is deduced in a similar way: namely, that if the number of values of a rational function of n arbitrary quantities be less than the greatest prime number which is itself not greater than n, the number of values of that function must then be either two or one.

[26.] It is a necessary consequence of the foregoing argument, that there must be a fallacy in the very ingenious process by which Mr. JERRARD has attempted to reduce the general equation of the fifth degree to the solvible form of DE MOIVRE, namely,

$$x^5 - 5bx^3 + 5b^2x - 2e = 0,$$

of which a root may be expressed as follows,

$$x = \sqrt[5]{\{e + \sqrt{e^2 - b^5}\}} + \frac{b}{\sqrt[5]{\{e + \sqrt{e^2 - b^5}\}}}:$$

because this process of reduction would, if valid, conduct to a finite (though complicated) expression for a root x of the general equation of the fifth degree,

$$x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5 = 0,$$

with five arbitrary coefficients, real or imaginary, as a function of those five coefficients, through the previous resolution of certain auxiliary equations below the fifth degree, namely, a cubic, two quadratics, another cubic, and a biquadratic, besides linear equations and DE MOIVRE's solvible form; and therefore ultimately through the extraction of a finite number of radicals, namely, a square-root, a cube-root, three square-roots, a cube-root, a square-root, a cube-root, three square-roots, and a fifth-root. Accordingly, the fallacy of this process of reduction has been pointed out by the writer of the present paper, in an "Inquiry into the Validity of a Method recently proposed by GEORGE B. JERRARD, Esq., for transforming and resolving Equations of Elevated Degrees:" undertaken at the request of the British Association for the Advancement of Science, and published in their Sixth Report. But the same Inquiry has confirmed the adequacy of Mr. JERRARD's method to accomplish an almost equally curious and unexpected transformation, namely, the reduction of the general equation of the fifth degree to the trinomial form

$$x^5 + Dx + E = 0;$$

and therefore ultimately to this very simple form

$$x^5 + x = e;$$

in which, however, it is essential to observe that e will in general be imaginary even when the original coefficients are real. If then we make, in this last form,

$$x = \rho(\cos\theta + \sqrt{-1}\sin\theta),$$

and

$$e = r(\cos v + \sqrt{-1}\sin v),$$

we can, with the help of Mr. JERRARD's method, reduce the general equation of the fifth degree, with five arbitrary and imaginary coefficients, to the system of the two following equations, which involve only real quantities:

$$\rho^5 \cos 5\theta + \rho\cos\theta = r\cos v; \quad \rho^5 \sin 5\theta + \rho\sin\theta = r\sin v;$$

in arriving at which system, the quantities r and v are determined, without tentation, by a finite number of rational combinations, and of extractions of square-roots and cube-roots of imaginaries, which can be performed by the help of the usual logarithmic tables; and ρ and θ may afterwards be found from r and v, by *two new tables of double entry*, which the writer of the present paper has had the curiosity to construct and to apply.

[27.] In general, if we change x to $x + \sqrt{-1}y$, and a_i to $a_i + \sqrt{-1}b_i$, the equation of the fifth degree becomes

$$(x + \sqrt{-1}y)^5 + (a_i + \sqrt{-1})(x + \sqrt{-1}y)^4 + \cdots + a_5 + \sqrt{-1}b_5 = 0,$$

and resolves itself into the two following:

$$\begin{aligned}\text{I.}\quad & x^5 - 10x^3y^2 + 5xy^4 + a_1(x^4 - 6x^2y^2 + y^4) - b_1(4x^3y - 4xy^3) \\ & + a_2(x^3 - 3xy^2) - b_2(3x^2y - y^3) + a_3(x^2 - y^2) - 2b_3xy + a_4x - b_4y + a_5 = 0;\end{aligned}$$

and

$$\begin{aligned}\text{II.}\quad & 5x^4y - 10x^2y^3 + y^5 + a_1(4x^3y - 4xy^3) + b_1(x^4 - 6x^2y^2 + y^4) \\ & + a_2(3x^2y - y^3) + b_2(x^3 - 3xy^2) + 2a_3xy + b^3(x^2 - y^2) + a_4y + b_4x + b_5 = 0;\end{aligned}$$

in which all the quantities are real: and the problem of resolving the general equation with imaginary coefficients is really equivalent to the problem of resolving this last system; that is, to the problem of deducing, from it, *two real functions* (x and y) of TEN *arbitrary real quantities* $a_1, \ldots a_5, b_1, \ldots b_5$. Mr. JERRARD has therefore accomplished a very remarkable simplification of this general problem, since he has reduced it to the problem of discovering *two real functions of* TWO *arbitrary real quantities*, by showing that, without any real loss of generality, it is permitted to suppose

$$a_1 = a_2 = a_3 = b_1 = b_2 = b_3 = b_4 = 0,$$

and

$$a_4 = 1,$$

a_5 and b_5 alone remaining arbitrary: though he has failed (as the argument developed in this paper might have shown beforehand that he must necessarily fail) in his endeavour to calculate the latter two, or the former ten functions, through any finite number of extractions of square-roots, cube-roots, and fifth-roots of expressions of the form $a + \sqrt{-1}b$.

[28.] But when we come to consider in what sense it is true that we are in possession of methods for extracting, without tentation, such roots of such imaginary expressions; and therefore in what sense we are permitted to postulate the extraction of such radicals, or the determination of both x and y, in an imaginary equation of the form

$$x + \sqrt{-1}y = \sqrt[\alpha]{a + \sqrt{-1}b},$$

as an instrument of calculation in algebra; we find that this depends ultimately on our being able to reduce all such extractions to the employment of *tables of single entry*: or, in more theoretical language, to *real functions of single real variables.* In fact, the equation last-mentioned gives

$$(x + \sqrt{-1}y)^\alpha = a + \sqrt{-1}b,$$

that is, it gives the system of the two following:

$$x^\alpha - \frac{\alpha(\alpha-1)}{1\,.\,2} x^{\alpha-2} y^2 + \&\text{c.} = a, \quad \alpha x^{\alpha-1} y - \frac{\alpha(\alpha-1)(\alpha-2)}{1\,.\,2\,.\,3} x^{\alpha-3} y^3 + \&\text{c.} = b;$$

which, again, give

$$(x^2 + y^2)^\alpha = a^2 + b^2,$$

and

$$\frac{\alpha \dfrac{y}{x} - \dfrac{\alpha(\alpha-1)(\alpha-2)}{1\,.\,2\,.\,3} \left(\dfrac{y}{x}\right)^3 + \cdots}{1 - \dfrac{\alpha(\alpha-1)}{1\,.\,2} \left(\dfrac{y}{x}\right)^2 + \cdots} = \frac{b}{a}.$$

If then we put

$$\phi_1(\rho) = \rho^\alpha,$$

and

$$\phi_2(\tau) = \frac{\alpha\tau - \dfrac{\alpha(\alpha-1)(\alpha-2)}{1\,.\,2\,.\,3} \tau^3 + \cdots}{1 - \dfrac{\alpha(\alpha-1)}{1\,.\,2} \tau^2 + \cdots};$$

and observe that these two real and rational functions ϕ_1 and ϕ_2 of single real quantities have always real inverses, ϕ_1^{-1} and ϕ_2^{-1}, at least if the operation ϕ_1^{-1} be performed on a positive quantity, while the function $\phi_1^{-1}(r^2)$ has but one real and positive value, and the function $\phi_2^{-1}(t)$ has α real values; we see that the determination of x and y in the equation

$$x + \sqrt{-1}y = \sqrt[\alpha]{a + \sqrt{-1}b},$$

comes ultimately to the calculation of the following real functions of single real variables, of which the inverse functions are rational:

$$x^2 + y^2 = \phi_1^{-1}(a^2 + b^2); \quad \frac{y}{x} = \phi_2^{-1}\left(\frac{b}{a}\right);$$

and to the extraction of a single real square-root, which gives

$$x = \pm \sqrt{\left\{ \frac{\phi_1^{-1}(a^2 + b^2)}{1 + \left(\phi_2^{-1} \dfrac{b}{a}\right)^2} \right\}},$$

$$y = \pm \left(\phi_2^{-1} \frac{b}{a}\right) \cdot \sqrt{\left\{ \frac{\phi_1^{-1}(a^2 + b^2)}{1 + \left(\phi_2^{-1} \dfrac{b}{a}\right)^2} \right\}}.$$

Now, notwithstanding the importance of those two particular forms of rational functions ϕ_1 and ϕ_2 which present themselves in separating the real and imaginary part of the radical $\sqrt[\alpha]{a + \sqrt{-1}b}$, and of which the former is a power of a single real variable, while the latter is the tangent of a multiple and real arc expressed in terms of the single and real arc corresponding; it may appear with reason that these functions do not possess such an eminent prerogative of simplicity as to entitle the *inverses of them alone* to be admitted into elementary algebra, to the exclusion of the inverses of all other real and rational functions of single real variables. And since the general equation of the fifth degree, with real or imaginary coefficients, has been reduced, by Mr. Jerrard's* method, to the system of the two real equations

$$x^5 - 10x^3y^2 + 5xy^4 + x = a, \quad 5x^4y - 10x^2y^3 + y^5 + y = b,$$

it ought, perhaps, to be now the object of those who interest themselves in the improvement of this part of algebra, to inquire whether the dependence of the two real numbers x and y, in these two last equations, on the two real numbers a and b, cannot be expressed by the help of the real inverses of some new real and rational, or even transcendental functions of single real variables; or, (to express the same thing in a practical, or in a geometrical form,) to inquire whether the two sought real numbers cannot be calculated by a finite number of tables of single entry, or constructed by the help of a finite number of curves: although the argument of Abel excludes all hope that this can be accomplished, if we confine ourselves to those particular forms of rational functions which are connected with the extraction of radicals.

It may be proper to state, that in adopting, for the convenience of others, throughout this paper, the usual language of algebraists, especially respecting real and imaginary quantities, the writer is not to be considered as abandoning the views which he put forward in his Essay on Conjugate Functions, and on Algebra as the Science of Pure Time, published in the second Part of the seventeenth volume of the Transactions of the Academy: which views he still hopes to develop and illustrate hereafter.

He desires also to acknowledge, that for the opportunity of reading the original argument of Abel, in the first volume of Crelle's Journal, he is indebted to the kindness of his

* *Mathematical Researches*, by George B. Jerrard, Esq., A.B.; printed by William Strong, Clare-street, Bristol.

friend Mr. LUBBOCK; and that his own remarks were written first in private letters to that gentleman, before they were thrown into the form of a communication to the Royal Irish Academy.

ADDITION

Since the foregoing paper was communicated, the writer has seen, in the first Part of the Philosophical Transactions for 1837, an essay entitled "Analysis of the Roots of Equations," by a mathematician of a very high genius, the Rev. R. MURPHY, Fellow of Caius College, Cambridge; who appears to have been led, by the analogy of the expressions for roots of equations of the first four degrees, to conjecture that the five roots $x_1\, x_2\, x_3\, x_4\, x_5$ of the general equation of the fifth degree,

$$x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0, \tag{1}$$

can be expressed as finite irrational functions of the five arbitrary coefficients a, b, c, d, e, as follows:

$$\left.\begin{aligned}
x_1 &= \frac{-a}{5} + \sqrt[5]{\alpha} + \sqrt[5]{\beta} + \sqrt[5]{\gamma} + \sqrt[5]{\delta},\\
x_2 &= \frac{-a}{5} + \omega\sqrt[5]{\alpha} + \omega^2\sqrt[5]{\beta} + \omega^3\sqrt[5]{\gamma} + \omega^4\sqrt[5]{\delta},\\
x_3 &= \frac{-a}{5} + \omega^2\sqrt[5]{\alpha} + \omega^4\sqrt[5]{\beta} + \omega\sqrt[5]{\gamma} + \omega^3\sqrt[5]{\delta},\\
x_4 &= \frac{-a}{5} + \omega^3\sqrt[5]{\alpha} + \omega\sqrt[5]{\beta} + \omega^4\sqrt[5]{\gamma} + \omega^2\sqrt[5]{\delta},\\
x_5 &= \frac{-a}{5} + \omega^4\sqrt[5]{\alpha} + \omega^3\sqrt[5]{\beta} + \omega^2\sqrt[5]{\gamma} + \omega\sqrt[5]{\delta},
\end{aligned}\right\} \tag{2}$$

ω being an imaginary fifth-root of unity, and $\alpha\ \beta\ \gamma\ \delta$ being the four roots of an auxiliary biquadratic equation,

$$\left.\begin{aligned}
\alpha &= \alpha' + \sqrt{\beta'} + \sqrt{\gamma'} + \sqrt{\delta'},\\
\beta &= \alpha' + \sqrt{\beta'} - \sqrt{\gamma'} - \sqrt{\delta'},\\
\gamma &= \alpha' - \sqrt{\beta'} + \sqrt{\gamma'} - \sqrt{\delta'},\\
\delta &= \alpha' - \sqrt{\beta'} - \sqrt{\gamma'} + \sqrt{\delta'};
\end{aligned}\right\} \tag{3}$$

in which $\beta'\ \gamma'\ \delta'$ are the three roots of the auxiliary cubic equation,

$$\left.\begin{aligned}
\beta' &= \alpha'' + \sqrt[3]{\beta''} + \sqrt[3]{\gamma''},\\
\gamma' &= \alpha'' + \theta\sqrt[3]{\beta''} + \theta^2\sqrt[3]{\gamma''},\\
\delta' &= \alpha'' + \theta^2\sqrt[3]{\beta''} + \theta\sqrt[3]{\gamma''};
\end{aligned}\right\} \tag{4}$$

θ being an imaginary cube-root of unity, and β'', γ'' being the two roots of an auxiliary quadratic,

$$\left.\begin{aligned}
\beta'' &= \alpha''' + \sqrt{\alpha^{\mathrm{IV}}},\\
\gamma'' &= \alpha''' - \sqrt{\alpha^{\mathrm{IV}}}.
\end{aligned}\right\} \tag{5}$$

And, doubtless, it is allowed to represent any five arbitrary quantities x_1 x_2 x_3 x_4 x_5 by the system of expressions (2) (3) (4) (5), in which a, ω and θ are such that

$$a = -(x_1 + x_2 + x_3 + x_4 + x_5), \tag{6}$$

$$\omega^4 + \omega^3 + \omega^2 + \omega + 1 = 0, \tag{7}$$

$$\theta^2 + \theta + 1 = 0, \tag{8}$$

provided that the auxiliary quantities α β γ δ α' β' γ' δ' α'' β'' γ'' α''' α^{IV} be determined so as to satisfy the conditions

$$\left.\begin{aligned} 5\sqrt[5]{\alpha} &= x_1 + \omega^4 x_2 + \omega^3 x_3 + \omega^2 x_4 + \omega x_5, \\ 5\sqrt[5]{\beta} &= x_1 + \omega^3 x_2 + \omega x_3 + \omega^4 x_4 + \omega^2 x_5, \\ 5\sqrt[5]{\gamma} &= x_1 + \omega^2 x_2 + \omega^4 x_3 + \omega x_4 + \omega^3 x_5, \\ 5\sqrt[5]{\delta} &= x_1 + \omega x_2 + \omega^2 x_3 + \omega^3 x_4 + \omega^4 x_5, \end{aligned}\right\} \tag{9}$$

$$\left.\begin{aligned} 4\alpha' &= \alpha + \beta + \gamma + \delta, \\ 4\beta' &= \alpha + \beta - \gamma - \delta, \\ 4\gamma' &= \alpha - \beta + \gamma - \delta, \\ 4\delta' &= \alpha - \beta - \gamma + \delta, \end{aligned}\right\} \tag{10}$$

$$\left.\begin{aligned} 3\alpha'' &= \beta' + \gamma' + \delta', \\ 3\sqrt[3]{\beta''} &= \beta' + \theta^2\gamma' + \theta\delta', \\ 3\sqrt[3]{\gamma''} &= \beta' + \theta\gamma' + \theta^2\delta', \end{aligned}\right\} \tag{11}$$

$$\left.\begin{aligned} 2\alpha''' &= \beta'' + \gamma'', \\ 2\sqrt{\alpha^{\mathrm{IV}}} &= \beta'' - \gamma''. \end{aligned}\right\} \tag{12}$$

But it is not true that the four auxiliary quantities α', α'', α''', α^{IV}, determined by these conditions, are symmetric functions of the five quantities x_1, x_2, x_3, x_4, x_5, or rational functions of a, b, c, d, e, as Mr. Murphy appears to have conjectured them to be.

In fact, the conditions just mentioned give, in the first place, expressions for α, β, γ, δ, α', as functions of the five roots x_1, x_2, x_3, x_4, x_5, which functions are rational and integral and homogeneous of the fifth dimension; they give, next, expressions for β', γ', δ', α'', as functions of the tenth dimension; for β'', γ'', α''', of the thirtieth; and for α^{IV}, of the sixtieth dimension. And Mr. Murphy has rightly remarked that the function α^{IV} may be put under the form

$$\alpha^{\mathrm{IV}} = k\mathrm{A}_1^2 \,.\, \mathrm{A}_2^2 \,.\, \mathrm{A}_3^2 \,.\, \mathrm{A}_4^2 \,.\, \mathrm{A}_5^2 \,.\, \mathrm{A}_6^2 \,.\, \mathrm{B}_1^2 \ldots \mathrm{B}_6^2 \,.\, \mathrm{C}_1^2 \ldots \mathrm{C}_6^2 \,.\, \mathrm{D}_1^2 \ldots \mathrm{D}_6^2 \,.\, \mathrm{E}_1^2 \ldots \mathrm{E}_6^2, \tag{13}$$

in which k is a numerical constant, and

$$\left.\begin{aligned} \mathrm{A}_1 &= x_2 - x_4 + \omega(x_3 - x_4) + \omega^2(x_3 - x_5), \\ \mathrm{A}_2 &= x_3 - x_2 + \omega(x_5 - x_2) + \omega^2(x_5 - x_4), \\ \mathrm{A}_3 &= x_4 - x_5 + \omega(x_2 - x_5) + \omega^2(x_2 - x_3), \\ \mathrm{A}_4 &= x_5 - x_3 + \omega(x_4 - x_3) + \omega^2(x_4 - x_2), \end{aligned}\right\} \tag{14}$$

$$\left.\begin{aligned} \mathrm{A}_5 &= x_2 - x_5 + (\omega^2+\omega^3)(x_3-x_4), \\ \mathrm{A}_6 &= x_3 - x_4 + (\omega^2+\omega^3)(x_5-x_2); \end{aligned}\right\} \tag{15}$$

these six being the only linear factors of $\sqrt{\dfrac{\alpha^{\mathrm{IV}}}{k}}$ which do not involve x_1. But the expression (14) give, by (7),

$$\begin{aligned} &\left(\frac{\omega}{1+\omega}\right)^2 \mathrm{A}_1\mathrm{A}_2\mathrm{A}_3\mathrm{A}_4 \\ &\quad = \{x_2^2+x_3^2+x_4^2+x_5^2-(x_2+x_5)(x_3+x_4)\}^2 \\ &\quad\quad +\{(x_2-x_5)^2+(x_2-x_4)(x_5-x_3)\}\{(x_3-x_4)^2+(x_2-x_3)(x_5-x_4)\}; \end{aligned} \tag{16}$$

and the expressions (15) give

$$\frac{\omega^3}{1+\omega}\mathrm{A}_5\mathrm{A}_6 = (x_3-x_4)^2+(x_2-x_5)(x_3-x_4)-(x_2-x_5)^2; \tag{17}$$

the part of α^{IV}, which is of highest dimension relatively to x_1, is therefore of the form

$$\begin{aligned} &\mathrm{N}x_1^{48}(\{x_2^2+x_3^2+x_4^2+x_5^2-(x_2+x_5)(x_3+x_4)\}^2 \\ &\quad + \{(x_2-x_5)^2+(x_2-x_4)(x_5-x_3)\}\{(x_3-x_4)^2+(x_2-x_3)(x_5-x_4)\})^2 \\ &\quad \times \{(x_3-x_4)^2+(x_2-x_5)(x_3-x_4)-(x_2-x_5)^2\}^2, \end{aligned} \tag{18}$$

N being a numerical coefficient; and consequently the coefficients, in α^{IV}, of the products $x_1^{48}x_2^{11}x_3$ and $x_1^{48}x_2x_3^{11}$ are, respectively $-6\mathrm{N}$ and $-4\mathrm{N}$; they are therefore unequal, and α^{IV} is not a symmetric function of x_1, x_2, x_3, x_4, x_5.

The same defect of symmetry may be more easily proved for the case of the function α', by observing that when x_1 and x_5 are made $= 0$, the expression

$$\begin{aligned} 4.5^5 \,.\, \alpha' = {} & (x_1+\omega x_2+\omega^2x_3+\omega^3x_4+\omega^4x_5)^5 \\ & +(x_1+\omega^2x_2+\omega^4x_3+\omega x_4+\omega^3x_5)^5 \\ & +(x_1+\omega^3x_2+\omega x_3+\omega^4x_4+\omega^2x_5)^5 \\ & +(x_1+\omega^4x_2+\omega^3x_3+\omega^2x_4+\omega x_5)^5 \end{aligned} \tag{19}$$

becomes

$$\begin{aligned} &(x_2+\omega x_3+\omega^2x_4)^5+(x_2+\omega^2x_3+\omega^4x_4)^5+(x_2+\omega^3x_3+\omega x_4)^5+(x_2+\omega^4x_3+\omega^3x_4)^5 \\ &\quad = 4x_2^5-5x_2^4(x_3+x_4)-10x_2^3(x_3^2+2x_3x_4+x_4^2)-10x_2^2(x_3^3+3x_3^2x_4-12x_3x_4^2+x_4^3) \\ &\quad\quad -5x_2(x_3^4-16x_3^3x_4+6x_3^2x_4^2+4x_3x_4^3+x_4^4) \\ &\quad\quad +4x_3^5-5x_3^4x_4-10x_3^3x_4^2-10x_3^2x_4^3-5x_3x_4^4+4x_4^5, \end{aligned} \tag{20}$$

which is evidently unsymmetric.

The elegant analysis of Mr. Murphy fails therefore to establish any conclusion opposed to the argument of Abel.