ON THE SOLUTION OF THE EQUATION OF LAPLACE'S FUNCTIONS

By

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Rev. Professor Graves communicated the following extract from a letter addressed to him (under date of January 26th, 1855) by Sir William R. Hamilton:—

"MY DEAR GRAVES,—You may like, perhaps, to see a way in which I have to-day, for my own satisfaction, confirmed (not that they required confirmation) some of the results announced by you to the Academy on Monday evening last.

"Let us then consider the function (suggested by you),

$$\Sigma i^l j^m k^n = (l, m, n) i^l h^m k^n; \tag{1}$$

where l, m, n are positive and integer exponents (0 included); the summation Σ refers to all the possible arrangements of the $l + m + n$ factors, whereof the number is

$$N_{l,m,n} = \frac{(l + m + n)!}{l!\, m!\, n!}; \tag{2}$$

each of these N arrangements gives (by the rules of $i\,j\,k$) a product $= \pm 1 \,.\, i^l j^m k^n$; and the sum of these positive or negative unit-coefficients, ± 1, thus obtained, is the numerical coefficient denoted by (l, m, n).

"Since each arrangement must have i or j or k to the left, we may write,

$$\Sigma i^l j^m k^n = i\, \Sigma i^{l-1} j^m k^n + j\, \Sigma i^l j^{m-1} k^n + k\, \Sigma i^l j^m k^{n-1}; \tag{3}$$

and it is easy to see that the coefficient (l, m, n), or the sum $\Sigma(\pm 1)$, vanishes, if *more than one* of the exponents, l, m, n, be *odd*. Assume, therefore, as a new notation,

$$(2\lambda, 2\mu, 2\nu) = \{\lambda, \mu, \nu\}; \tag{4}$$

which will give, by (3), and by the principle last mentioned respecting odd exponents,

$$\begin{aligned} (2\lambda + 1, 2\mu, 2\nu) &= \{\lambda, \mu, \nu\}; \\ (2\lambda - 1, 2\mu, 2\nu) &= \{\lambda - 1, \mu, \nu\}. \end{aligned} \tag{5}$$

We shall then have, by the mere notation,

$$\Sigma i^{2\lambda} j^{2\mu} k^{2\nu} = \{\lambda, \mu, \nu\} i^{2\lambda} j^{2\mu} k^{2\nu}; \tag{6}$$

and, by treating this equation on the plan of (3),

$$\{\lambda, \mu, \nu\} = \{\lambda - 1, \mu, \nu\} + \{\lambda, \mu - 1, \nu\} + \{\lambda, \mu, \nu - 1\}. \tag{7}$$

By a precisely similar reasoning, attending only to j and k, or making $\lambda = 0$, we have an expresssion of the form,

$$\Sigma j^{2\mu} k^{2\nu} = \{\mu, \nu\} j^{2\mu} k^{2\nu}, \tag{8}$$

where the coefficients $\{\mu, \nu\}$ must satisfy the analogous equation in differences,

$$\{\mu, \nu\} = \{\mu - 1, \nu\} + \{\mu, \nu - 1\}, \tag{9}$$

together with the initial conditions,

$$\{\mu, 0\} = 1, \quad \{0, \nu\} = 1. \tag{10}$$

Hence, it is easy to infer that

$$\{\mu, \nu\} = \frac{(\mu + \nu)!}{\mu!\, \nu!}; \tag{11}$$

one way of obtaining which result is, to observe that the generating function has the form,

$$\Sigma \{\mu, \nu\} u^\mu v^\nu = (1 - u - v)^{-1}. \tag{12}$$

In like manner, if we combine the equation in differences (7), with the initial conditions derived from the foregoing solution of a less complex problem, namely, with

$$\{0, \mu, \nu\} = \{\mu, \nu\}, \quad \{\lambda, 0, \nu\} = \{\lambda, \nu\}, \quad \{\lambda, \mu, 0\} = \{\lambda, \mu\}, \tag{13}$$

when the second members are interpreted as in (11), we find that the (slightly) more complex generating function sought is,

$$\Sigma \{\lambda, \mu, \nu\} t^\lambda u^\mu v^\nu = (1 - t - u - v)^{-1}; \tag{14}$$

and therefore that the required form of the coefficient is,

$$\{\lambda, \mu, \nu\} = \frac{(\lambda + \mu + \nu)!}{\lambda!\, \mu!\, \nu!}; \tag{15}$$

as, I have no doubt, you had determined it to be.

"With the same signification of { }, we have, by (2),

$$N_{l,m,n} = \{l, m, n\}; \tag{16}$$

therefore, dividing Σ by N, or the *sum* by the *number*, we obtain, as an expression for what you happily call the MEAN VALUE of the product $i^{2\lambda} j^{2\mu} k^{2\nu}$ the following:

$$M i^{2\lambda} j^{2\mu} k^{2\nu} = \frac{\{\lambda, \mu, \nu\}}{\{2\lambda, 2\mu, 2\nu\}} i^{2\lambda} j^{2\mu} k^{2\nu}; \tag{17}$$

or, substituting for { } its value (15), and writing for abridgement

$$\kappa = \lambda + \mu + \nu, \tag{18}$$

$$Mi^{2\lambda} j^{2\mu} k^{2\nu} = \frac{(-1)^\kappa \kappa! (2\lambda)!\,(2\mu)!\,(2\nu)!}{(2\kappa)!\,\lambda!\,\mu!\,\nu!}. \tag{19}$$

In like manner,

$$Mi^{2\lambda+1} j^{2\mu} k^{2\nu} = \frac{i(-1)^\kappa \kappa!\,(2\lambda+1)!\,(2\mu)!\,(2\nu)!}{(2\kappa+1)!\,\lambda!\,\mu!\,\nu!}. \tag{20}$$

"The whole theory of what you call the *mean values*, of *products* of positive and integer *powers* of $i\,j\,k$, being contained in the foregoing remarks, let us next apply it to the determination of the mean value of a *function* of $x + iw$, $y + jw$, $z + kw$; or, in other words, let us investigate the equivalent for your

$$Mf(x + iw,\, y + jw,\, z + kw): \tag{21}$$

by developing this function f according to ascending powers of w, and by substituting, for every product of powers of $i\,j\,k$, its *mean* value determined as above. Writing, as you propose,

$$\frac{d}{dw} = D, \quad \frac{d}{dx} = D_1, \quad \frac{d}{dy} = D_2, \quad \frac{d}{dz} = D_3, \tag{22}$$

we are to calculate and to sum the general term of (21), namely,

$$Mi^l j^m k^n \times \frac{w^{l+m+n}}{l!\,m!\,n!} D_1^l D_2^m D_3^n f(x, y, z). \tag{23}$$

One only of the exponents l, m, n, can usefully be *odd*, by properties of the *mean* function, which have been already stated. If *all* be *even*, and if we make

$$l = 2\lambda, \quad m = 2\mu, \quad n = 2\nu, \tag{24}$$

the corresponding part of the general term of Mf, namely, the part independent of $i\,j\,k$, is by (15), (18), (19),

$$\frac{(-w^2)^\kappa}{(2\kappa)!} \{\lambda, \mu, \nu\} D_1^{2\lambda} D_2^{2\mu} D_3^{2\nu} f(x, y, z); \tag{25}$$

whereof the sum, relatively to λ, μ, ν, when *their* sum κ is given, is,

$$\frac{(-w^2)^\kappa}{(2\kappa)!} (D_1^2 + D_2^2 + D_3^2)^\kappa f(x, y, z) = \frac{(w \lhd)^{2\kappa}}{(2\kappa)!} f(x, y, z), \tag{26}$$

if my signification of $\lhd$ be adopted, so that

$$\lhd = iD_1 + jD_2 + kD_3; \tag{27}$$

and another summation, performed on (26), with respect to κ, gives, for the part of Mf which is independent of $i\,j\,k$, the expression,

$$\tfrac{1}{2}(e^{w\,\triangleleft} + e^{-w\,\triangleleft})f(x,y,z). \tag{28}$$

"Again, by supposing, in (23),

$$l = 2\lambda + 1, \quad m = 2\mu, \quad n = 2\nu, \tag{29}$$

and by attending to (20), we obtain the term,

$$\frac{wiD_1(-w^2)^\kappa}{(2\kappa+1)!}\{\lambda,\mu,\nu\}D_1^{2\lambda}D_2^{2\mu}D_3^{2\nu}f(x,y,z). \tag{30}$$

Adding the two other general terms correspondent, in which iD_1 is replaced by jD_2 and by kD_3, we change iD_1 to $\triangleleft$; and obtain, by a first summation, the term

$$\frac{(w\,\triangleleft)^{2\kappa+1}}{(2\kappa+1)!}f(x,y,z); \tag{31}$$

and, by a second summation, we obtain

$$\tfrac{1}{2}(e^{w\,\triangleleft} - e^{-w\,\triangleleft})f(x,y,z), \tag{32}$$

as the *part* of the mean function Mf, which involves expressly $i\,j\,k$. Adding the two parts, (28) and (32), we are conducted finally to the very simple and remarkable transformation of the MEAN FUNCTION Mf, of which the discovery is due to you:

$$Mf(x+iw, y+jw, z+kw) = e^{w\,\triangleleft}f(x,y,z). \tag{33}$$

In like manner,

$$M\phi(x-iw, y-jw, z-kw) = e^{-w\,\triangleleft}\phi(x,y,z). \tag{34}$$

Each of these two means of arbitrary functions, and therefore also their sum, is thus a value of the expression

$$(D^2 - \triangleleft^2)^{-1}0; \tag{35}$$

that is, the partial differential equation,

$$(D^2 + D_1^2 + D_2^2 + D_3^2)V = 0, \tag{36}$$

has its general integral, with two arbitary functions, f and ϕ, expressible as follows:

$$V = Mf(x+iw, y+jw, z+kw) + M\phi(x-iw, y-jw, z-kw); \tag{37}$$

which is another of your important results. You remarked that if the second member of the equation (36) had been U, the expression for V would contain the additional term,

$$e^{w\,\triangleleft}D^{-1}e^{-2w\,\triangleleft}D^{-1}e^{w\,\triangleleft}U. \tag{38}$$

In fact,

$$D + \lhd = e^{-w \lhd} d e^{w \lhd}, \quad D - \lhd = e^{w \lhd} d e^{-w \lhd}, \tag{39}$$

and therefore,

$$(D - \lhd)^{-1}(D + \lhd)^{-1} = e^{w \lhd} D^{-1} e^{-2w \lhd} D^{-1} e^{w \lhd}. \tag{40}$$

"Most of this letter is merely a repetition of your remarks, but the analysis employed may perhaps not be in all respects identical with yours: a point on which I shall be glad to be informed.

"I remain faithfully yours,
"WILLIAM ROWAN HAMILTON

"The Rev. Charles Graves, D. D."