ON THE INTEGRATIONS OF CERTAIN EQUATIONS

By

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Sir William Rowan Hamilton.

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Rev. Dr. Graves read a note from Sir W. R. Hamilton, in which he stated that he had lately arrived at a variety of results respecting the integrations of certain equations, which might not be unworthy of the acceptance of the Academy, and the investigation of which had been suggested to him by Mr. Carmichael's printed Paper, and by a manuscript which he had lent Sir W. Hamilton, who writes,—"In our considerations we do not quite agree, but I am happy to acknowledge my obligations to his writings for the suggestions above alluded to, as I shall hereafter more fully express.

"So long ago as 1846, I communicated to the Royal Irish Academy a transformation which may be written thus (see Proceedings for the July of that year):

$$D_x^2 + D_y^2 + D_z^2 = -(iD_x + jD_y + kD_z)^2; \tag{1}$$

and which was obviously connected with the celebrated equation of Laplace.

"But it had quite escaped my notice that the principles of quaternions allow also this other transformation, which Mr. Carmichael was the first to point out:

$$D_x^2 + D_y^2 + D_z^2 = (D_z - iD_x - jD_y)(D_z + iD_x + jD_y). \tag{2}$$

And therefore I had, of course, not seen, what Mr. Carmichael has since shown, that the integration of Laplace's equation of the *second* order may be made to depend on the integrations of *two linear* and conjugate equations, of which one is

$$(D_z - iD_x - jD_y)V = 0. \tag{3}$$

"I am disposed, for the sake of reference, to call this '*Carmichael's Equation*;' and have had the pleasure of recently finding its integral, under a form, or rather forms, so general as to extend even to *bi*quaternions.

"One of these forms is the following:*

$$V_{xyz} = e^{z(iD_x + jD_y)} V_{xy0}. \tag{4}$$

* "*Note, added during printing.*—Since writing the above, I have convinced myself that Mr. Carmichael had been in full possession of the exponential form of the integral, and probably also of my chief transformations thereof; although he seems to have chosen to put forward more prominently certain other forms, to which I have found objections, arising out of the non-commutative character of the symbols $i\,j\,k$ as factors, and on which forms I believe that he does not now insist.—W. R. H."

"Another is

$$V_{xyz} = (D_z + iD_x + jD_y) \int_0^z \cos\{z(D_x^2 + D_y^2)^{\frac{1}{2}}\} V_{xy0}\, dz; \tag{5}$$

where V_{xy0} is generally an *initial biquaternion*; and where the *single* definite integral admits of being usefully put under the form of a *double definite integral*, exactly analogous to, and (when we proceed to Laplace's equation) reproducing, a well known expression of Poisson's, to which Mr. Carmichael has referred.

"These specimens may serve to show to the Academy that I have been aiming to collect materials for future communications to their Transactions."