ON THE APPLICATION OF THE METHOD OF QUATERNIONS TO SOME DYNAMICAL QUESTIONS

By

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The following is the substance of the communications made to the Academy by Sir William Hamilton, on the application of the method of Quaternions to some dynamical questions:

The author stated that, during a visit which he had lately made to England, Sir John Herschel suggested to him that the internal character (if it may be so called) of the method of quaternions, or of vectors, as applied to algebraic geometry,—that character by which it is independent of any foreign and arbitrary axes of coordinates,—might make it useful in researches respecting the attractions of a system of bodies. A beginning of such a research had been made by Sir William Hamilton in October, 1844, which went so far, but only so far, as the deducing of the constancy of the plane of an orbit, and the equable description of areas, under one common formula, namely, the following:

$$\rho \frac{\mathrm{d}\rho}{\mathrm{d}t} - \frac{\mathrm{d}\rho}{\mathrm{d}t}\rho = \text{const.},$$

from the general expression of a central force, namely, from the equation

$$\rho \frac{\mathrm{d}^2\rho}{\mathrm{d}t^2} - \frac{\mathrm{d}^2\rho}{\mathrm{d}t^2}\rho = 0,$$

which asserts merely the *coaxality* of the vector ρ and the force $\dfrac{\mathrm{d}^2\rho}{\mathrm{d}t^2}$, or the existence of one common line along which this vector and this force are (similarly or oppositely) directed.

Since the suggestion above acknowledged was made, Sir William Hamilton has proposed to himself to express by an equation, on the principles of the method of vectors, the problem of any number of bodies attracting according to Newton's law: and has arrived at the formula

$$\frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2} = \Sigma \frac{m + \Delta m}{-\Delta\alpha\sqrt{(-\Delta\alpha^2)}}; \tag{A}$$

which may also be thus written,

$$\frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2} = \Sigma \frac{m'}{(\alpha - \alpha')\sqrt{\{-(\alpha - \alpha')^2\}}}; \tag{B}$$

and from which he has deduced anew the known laws of the centre of gravity, of areas, and of the *vis viva*, under the forms:

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^2 \Sigma \,.\, m\alpha = 0; \tag{C}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Sigma \,.\, m\left(\alpha\frac{\mathrm{d}\alpha}{\mathrm{d}t} - \frac{\mathrm{d}\alpha}{\mathrm{d}t}\alpha\right) = 0; \tag{D}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Sigma \,.\, \frac{m}{2}\left(\frac{\mathrm{d}\alpha}{\mathrm{d}t}\right)^2 + \frac{\mathrm{d}}{\mathrm{d}t}\Sigma \,.\, \frac{mm'}{\sqrt{\{-(\alpha-\alpha')^2\}}} = 0. \tag{E}$$

α is the vector and m the mass of one body; α' and m' of another; Σ sums for the system; t is the time, d the characteristic of differentiation; Δ (where used) is the mark of finite differencing.

To illustrate the method of treating equations of such forms as these, let us consider briefly the problem of *two* bodies, or of *one* body, as it presents itself, in the method of quaternions, with Newton's law of attraction, coordinates being not employed. The differential equation may be thus written,

$$\frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2} = \frac{\mathrm{M}}{\alpha\sqrt{(-\alpha^2)}}; \tag{1}$$

α being the vector of the attracted body, drawn from the attracting one; t the time, d the mark of differentiation; and M the attracting mass, or the sum of the two such masses. This equation gives

$$\alpha\frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2} - \frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2}\alpha = 0; \tag{2}$$

which expresses merely that the force is central; and gives by integration a result already alluded to (as independent of that function of the distance which enters into the law of attraction), namely,

$$\frac{\alpha}{2}\frac{\mathrm{d}\alpha}{\mathrm{d}t} - \frac{\mathrm{d}\alpha}{\mathrm{d}t}\frac{\alpha}{2} = \beta; \quad \mathrm{d}\beta = 0; \tag{3}$$

the constant β being a new vector, perpendicular in direction to the plane of the orbit, and in magnitude representing the double of the areal velocity, which velocity is thus seen to be constant, as also is the plane. For we have at once, by (3),

$$\alpha\beta + \beta\alpha = 0, \tag{4}$$

implying that the variable vector α is perpendicular to the constant vector β; and also

$$\int(\alpha \,.\, \mathrm{d}\alpha - \mathrm{d}\alpha \,.\, \alpha) = 2\beta(t - t_0), \tag{5}$$

if t_0 be the value of t at the commencement of the integral.

Make now, to distinguish between the length and direction of the vector,

$$\alpha = r\iota, \quad r = \sqrt{(-\alpha^2)}, \quad \iota^2 = -1; \tag{6}$$

we shall have

$$\mathrm{d}\alpha = r \,.\, \mathrm{d}\iota + \mathrm{d}r \,.\, \iota, \tag{7}$$

and because r and $\mathrm{d}r$ are scalar (or real) quantities,

$$\alpha \,.\, \mathrm{d}\alpha = r^2\iota \,.\, \mathrm{d}\iota - r \,.\, \mathrm{d}r, \quad \mathrm{d}\alpha \,.\, \alpha = r^2 \,\mathrm{d}\iota \,.\, \iota - r \,.\, \mathrm{d}r; \tag{8}$$

therefore

$$\beta \,.\, \mathrm{d}t = \tfrac{1}{2}(\alpha \,.\, \mathrm{d}\alpha - \mathrm{d}\alpha \,.\, \alpha) = \frac{r^2}{2}(\iota \,.\, \mathrm{d}\iota - \mathrm{d}\iota \,.\, \iota) = r^2\iota \,.\, \mathrm{d}\iota, \tag{9}$$

observing that the equation

$$\iota^2 = -1 \quad \text{gives} \quad \iota \,.\, \mathrm{d}\iota + \mathrm{d}\iota \,.\, \iota = 0. \tag{10}$$

The fundamental equation (1) of the problem becomes, by (6) and (9),

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}\alpha}{\mathrm{d}t} = \frac{\textsc{m}}{r^2\iota} = \frac{d\iota}{dt}\frac{\textsc{m}}{\beta}, \tag{11}$$

(in which last member the order of the factors is not indifferent), and therefore gives, by integration, since β as well as $\textsc{m}$ is constant,

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} - \iota\frac{\textsc{m}}{\beta} = \text{const.}; \tag{12}$$

or, as we may also write it,

$$\iota - \frac{\mathrm{d}\alpha}{dt}\frac{\beta}{\textsc{m}} = \epsilon, \quad \mathrm{d}\epsilon = 0. \tag{13}$$

We have, consequently, by (6) and (4),

$$\alpha\epsilon = -r - \alpha\frac{\mathrm{d}\alpha}{\mathrm{d}t}\frac{\beta}{\textsc{m}}, \quad \epsilon\alpha = -r + \frac{\mathrm{d}\alpha}{\mathrm{d}t}\alpha\frac{\beta}{\textsc{m}}; \tag{14}$$

and finally, by (3),

$$\alpha\epsilon + \epsilon\alpha + 2r = 2p, \tag{15}$$

if

$$p = \frac{-\beta^2}{\textsc{m}}, \tag{16}$$

the constant p being here not only a scalar by an essentially positive quantity, because the force is supposed to be attractive, or $\textsc{m} > 0$, while $\beta^2 < 0$. The equation (15) thus obtained, contains the law of elliptic, parabolic, or hyperbolic motion. For if we make (by way of comparison with known results),

$$\sqrt{(-\epsilon^2)} = e, \tag{17}$$

and

$$(\alpha, -\epsilon) = v, \tag{18}$$

$(\alpha, -\epsilon)$ denoting here the angle between the directions of α and ϵ, we have (by the formula (a) of the abstract of last November),

$$\alpha\epsilon + \epsilon\alpha = 2er\cos v; \tag{19}$$

and therefore, by (15),

$$r = \frac{p}{1 + e\cos v}, \tag{20}$$

which is the known equation of a conic section referred to a focus. The Greek letters, throughout, represent vectors: and the Italics, scalar quantities.

Suppose that we had no previous knowledge of the properties of cosines or of conics, we might have proceeded thus to investigate the nature of the locus represented by the equation (15). This locus is a *surface of revolution* round the line ϵ; because the differential of its equation being

$$\mathrm{d}\alpha\,.\,\epsilon + \epsilon\,.\,\mathrm{d}\alpha + 2\,\mathrm{d}r = 0 \tag{21}$$

if we cut it by a series of concentric spheres round the origin of vectors, the sections are contained in a series of planes perpendicular to ϵ; since

$$\mathrm{d}r = 0, \tag{22}$$

which is the differential equation of the first series, gives, by (21),

$$\mathrm{d}\alpha\,\epsilon + \epsilon\,\mathrm{d}\alpha = 0, \tag{23}$$

which is the differential equation of the second series. To study more closely this surface of revolution (15), make

$$\alpha = \gamma + \alpha', \tag{24}$$

γ being an arbitrary constant, and α' a variable vector; and since it must evidently give simpler and more symmetric results to suppose the vector γ co-axal with ϵ, than to make the contrary supposition, since we shall thus place the origin of the new vectors α' upon the axis of revolution of the surface, let

$$\epsilon\gamma - \gamma\epsilon = 0, \quad \text{or} \quad \gamma = g\epsilon, \tag{25}$$

g being an arbitrary scalar, to be disposed of according to convenience. Equations (24) and (25), combined with (6) and (17), will give, for every point of space,

$$-\alpha'^2 = -(\alpha - \gamma)^2 = r^2 + g^2\epsilon^2 + g(\alpha\epsilon + \epsilon\alpha); \tag{26}$$

and therefore, for every point of the locus (15),

$$-\alpha'^2 = r^2 - 2gr + g^2\epsilon^2 + 2gp. \tag{27}$$

The second member of this last equation may be made an exact square, by assuming

$$g^2e^2 + 2gp = g^2, \quad \text{that is,} \quad g = \frac{2p}{1-e^2} = 2a; \tag{28}$$

the scalar quotient

$$\frac{p}{1-e^2} = a, \quad \text{or the transformation} \quad p = a(1-e^2), \tag{29}$$

being thus suggested to our attention; and with this value of g we shall have, by (27),

$$-\alpha'^2 = (2a - r)^2, \tag{30}$$

that is,

$$2a = \sqrt{(-\alpha^2)} \pm \sqrt{(-\alpha'^2)}; \tag{31}$$

so that either the sum or the difference of the distances of any point on the locus (15) from the two *foci* of which the vectors are respectively 0 and $2a\epsilon$, is equal to the constant $2a$. It is not difficult to prove that the upper or the lower sign is to be taken, in the formula (31), according as e^2 is $<$ or $>$ 1. For the case $e^2 = 1$, the recent transformation fails.

Again, to find whether the locus has a *centre*, we may make

$$\alpha = \gamma' + \delta + g'\epsilon + \delta, \tag{32}$$

g' being a new disposable scalar, and δ a new variable vector; and, after having cleared the equation (15) of the radical r, or $\sqrt{(-\alpha^2)}$, by writing it as follows,

$$\alpha^2 + \left(p - \frac{\alpha\epsilon + \epsilon\alpha}{2}\right)^2 = 0, \tag{33}$$

we get

$$\begin{aligned} 0 &= (g'\epsilon + \delta)^2 + \left(p + g'e^2 - \frac{\delta\epsilon + \epsilon\delta}{2}\right)^2 \\ &= \delta^2 + \left(\frac{\delta\epsilon + \epsilon\delta}{2}\right)^2 + g''(\delta\epsilon + \epsilon\delta) + (p + g'e^2)^2 - g'^2e^2; \end{aligned} \tag{34}$$

if we make for abridgment

$$g'' = g' - (p + g'e^2). \tag{35}$$

If γ' or $g'\epsilon$ is to be the constant vector of the centre of the locus, it is necessary that to every variable vector, δ, which satisfies the equation (34), should correspond another vector $-\delta$, equal in length but opposite in direction, and satisfying the same equation; therefore the terms $g''(\delta\epsilon + \epsilon\delta)$ must disappear, and we we must have

$$g'' = 0, \quad g' = \frac{p}{1 - e^2} = a, \tag{36}$$

the constant a being thus suggested by the search after a centre, as well as by the search after a second focus. Making then $g' = a$ in (34), we find the following equation of the surface, when referred to its centre,

$$0 = \delta^2 + \left(\frac{\delta\epsilon + \epsilon\delta}{2}\right)^2 + ap; \tag{37}$$

in which

$$ap = a^2(1 - e^2) = a^2(1 + \epsilon^2). \tag{38}$$

And because in general, for any two vectors δ, ϵ, the following relation holds good,

$$\left(\frac{\delta\epsilon + \epsilon\delta}{2}\right)^2 = \left(\frac{\delta\epsilon - \epsilon\delta}{2}\right)^2 + \delta^2\epsilon^2, \tag{39}$$

we may write the equation (37) under the form

$$0 = (1 + \epsilon^2)(\delta^2 + a^2) + \left(\frac{\delta\epsilon - \epsilon\delta}{2}\right)^2. \tag{40}$$

This last equation shows that

$$\text{when} \quad \delta\epsilon - \epsilon\delta = 0, \quad \text{then} \quad \delta^2 + a^2 = 0; \tag{41}$$

that is to say, when δ is co-axal with, or parallel to ϵ, or, in other words, when the vector from the centre coincides (in either direction) with the axis of revolution of the surface, its length is $= \pm a$, according as a is $>$ or < 0.

The equation (37) shows that

$$\text{when} \quad \delta\epsilon + \epsilon\delta = 0, \quad \text{then} \quad \delta^2 + a^2(1 + \epsilon^2) = 0; \tag{42}$$

if therefore ϵ^2 be > -1, that is, if $e^2 < 1$, the length of every vector drawn from the centre perpendicularly to the axis of revolution will be

$$\sqrt{(-\delta^2)} = \alpha\sqrt{(1 - e^2)} = b, \tag{43}$$

b being a new scalar quantity; but if $e^2 > 1$, $\epsilon^2 < -1$, $1 + \epsilon^2 < 0$, then we shall have, by (42), *the absurd result of a* VECTOR δ *appearing to have a* POSITIVE SQUARE: whereas it is a first principle of the present method of calculation, *that the square of every vector* is to be regarded as a *negative number: which symbolical contradiction indicates the* GEOMETRICAL

IMPOSSIBILITY *of drawing from the centre to any point of the locus, a straight line which shall be perpendicular to the axis of revolution, in the case where* $e^2 > 1$. The locus has, in this case, two infinite branches enclosed within the two branches of the *asymptotic cone* which has for its equation

$$\delta^2 + \left(\frac{\delta\epsilon + \epsilon\delta}{2}\right)^2 = 0; \tag{44}$$

and nowhere penetrates within that *inscribed spheric surface*, which has for its equation

$$\delta^2 + a^2 = 0, \tag{45}$$

though it touches this last surface at the two points where it meets the axis of revolution. On the other hand, when $e^2 < 1$, the locus is entirely contained *within* the spheric surface (45), touching it, however, in like manner in two points upon the axis of revolution. A *finite* surface of revolution (the *ellipsoid*) might thus have been discovered, of which each point has a constant *sum* of distances from two fixed foci; and an *infinite* surface (the *hyperboloid*), with two separate sheets, of which each point has a constant *difference* of distances from two such foci: and all the other properties of these two surfaces of revolution might have been found, and may be proved anew, by pursuing this sort of analysis. A third distinct surface of the same class, but infinite in *one* direction only (the *paraboloid*), might have been suggested by the observation that the reduction to a centre fails in the case $e^2 = 1$, $\epsilon^2 = -1$. Its equation may be put under the form

$$(\epsilon\alpha'' - \alpha''\epsilon)^2 = 4p(\epsilon\alpha'' + \alpha''\epsilon), \tag{46}$$

by making

$$\alpha = \alpha'' - \frac{p\epsilon}{2}, \tag{47}$$

so that α'' is the vector from the vertex: and it lies entirely on one side of the plane which touches it at the vertex, namely, the plane

$$\epsilon\alpha'' + \alpha''\epsilon = 0. \tag{48}$$

In general whatever e or ϵ may be, and therefore for all the three surfaces, the length of the focal vector perpendicular to the axis is p; for, by (33), if we make

$$\epsilon\alpha + \alpha\epsilon = 0, \tag{49}$$

we get

$$\alpha^2 + p^2 = 0. \tag{50}$$

Indeed (15) then gives $r = p$.

Since

$$\alpha^2 + r^2 = 0, \quad \alpha \,.\, \mathrm{d}\alpha + \mathrm{d}\alpha \,.\, \alpha + 2r\,\mathrm{d}r = 0, \tag{51}$$

the differential equation (21) of the locus (15) may be put under the form

$$(r\epsilon - \alpha)\,\mathrm{d}\alpha + \mathrm{d}\alpha\,(r\epsilon - \alpha) = 0; \tag{52}$$

thus shewing that the vector $r\epsilon - \alpha$ is perpendicular to the differential d of the focal vector α, or that it is parallel to the normal to the locus, at the extremity of that focal vector. That normal, therefore, intersects the axis of revolution in a point, of which the focal vector is $r\epsilon$; the position of the normal is, therefore, entirely known, and every thing that depends upon it may be found, for the particular surfaces of revolution which have been here considered. For example, in the ellipsoid, the vector of the second focus, drawn from the first, has been seen to be $2a\epsilon$; if, then, we make

$$2a - r = r', \tag{53}$$

so that r' denotes the length of the second focal vector, drawn to the same point as the first focal vector, of which the length is r, we have $-r'\epsilon$ for the second focal vector of the intersection of the normal with this axis; the normal, therefore, cuts (internally) the interval between the two foci, into segments proportional to the two conterminous focal distances of the point upon the ellipsoid, and consequently bisects the angle between those focal distances. Again, if we divide the expression $r\epsilon - \alpha$ by the scalar quantity r, and multiply the quotient by a, we find that $\epsilon - \iota$ and $a\epsilon - a\iota$ are also expressions for vectors in the normal direction; and because $a\epsilon$ is the focal vector of the centre, while $-a\iota$ is a radius of the circumscribed sphere, opposite in direction to the focal vector of the point upon the ellipsoid, we see that if the focal vector of the extremity of this radius of the sphere be prolonged through the focus, it will cut perpendicularly the tangent plane to the ellipsoid. Again, the expression

$$\tau = a(\epsilon + \iota) - \alpha = (a - r)\iota + a\epsilon, \tag{54}$$

is easily seen to denote here a vector perpendicular to $\alpha - r\epsilon$, and therefore to the normal, because

$$\alpha\tau + \tau\alpha = r(\varepsilon\tau + \tau\varepsilon) = 2(ap - rr'); \tag{55}$$

but τ is also in the same plane with α and ϵ, and therefore is a vector parallel to the tangent to the elliptic section of the locus made by a plane passing through the axis of revolution; $a\iota$ is therefore the central vector of a point upon this tangent, because $\alpha - a\epsilon$ is the central vector of the point of contact; and the central vector of the second focus being $a\epsilon$, we have $a\iota - a\epsilon$ as an expression for the second focal vector of the same point upon the tangent; this second focal vector is therefore parallel to the normal, because $\iota - \epsilon$ is parallel thereto, and, consequently, it is the perpendicular let fall from the second focus on the tangent line or plane: and the foot of this perpendicular is thus seen to be at the extremity of that radius of the circumscribed circle or sphere, which is drawn in a direction similar (and not, as lately, opposite) to the direction of the first focal vector of the point on the ellipse or ellipsoid. We see, at the same time, that $-\tau$ is a symbol for the projection of the second focal vector upon the tangent line or plane; from which we may infer, by (55), that the product of the lengths of the two projections of the two focal vectors on the tangent is $= rr' - ap$, and therefore that it is less than the product rr' of the lengths of those two vectors by the constant quantity ap, or b^2, which constant must thus be equal to the product of the lengths of the projections of the same two vectors on the normal, so that we may write the equation

$$\mathrm{PP}' = ap = b^2, \tag{56}$$

if P and P' denote the lengths of the perpendiculars let fall from the two foci on the tangent, while b is the axis minor of the ellipse. Analogous reasoning may be applied to the hyperbola, or to the surface formed by its revolution round its transverse axis. Most of the foregoing geometrical results are well known, and probably all of them are so: but it may be considered worth while to have briefly indicated the manner in which they reproduce themselves in these new processes of calculation.

The vector drawn from the focus first considered to any arbitrary point upon the normal, may be represented by the expression

$$\nu = (1-n)\alpha + nr\epsilon, \tag{57}$$

in which n is an arbitrary scalar; and if this normal intersect another normal infinitely near it, then we may write, as the expression of this relation,

$$0 = \mathrm{d}\nu = (1-n)\mathrm{d}\alpha + n\epsilon\,\mathrm{d}r + (r\epsilon - \alpha)\,\mathrm{d}n : \tag{58}$$

comparing which differential equation with the forms (52) and (21) of the differential equation of the surface of revolution (15), we can eliminate the scalar differential dn, and deduce for n itself the expression

$$n = \frac{\mathrm{d}\alpha^2}{\mathrm{d}r^2 + \mathrm{d}\alpha^2}. \tag{59}$$

One way of satisfying these conditions is to suppose

$$n = 1, \quad \mathrm{d}r = 0, \quad \nu = r\epsilon; \tag{60}$$

which comes to considering the intersection of the given normal with the axis, and therefore with the other normal from points of the same generating circle of the surface of revolution: and this intersection is accordingly one centre of curvature of the surface. The only other way of obtaining an intersection of two normals infinitely near, is to suppose, by (58), the element dα coplanar with α and ϵ, or to pass to consecutive normals contained in the same plane drawn through the axis; that is to say, the other centre of curvature of the surface is the centre of curvature of its meridian. The length of the element of this meridian, that is the length of dα, is denoted by the radical $\sqrt{(-\mathrm{d}\alpha^2)}$, because the differential dα is a vector; and the length of the projection of this element on the focal vector is $\pm dr = \sqrt{(+\mathrm{d}r^2)}$, because dr is a scalar differential: therefore the length of the projection of the same element on a line perpendicular to the focal vector, and drawn in the plane through the axis, is denoted by this other radical, $\sqrt{(-\mathrm{d}\alpha^2 - \mathrm{d}r^2)}$; but the length of this last projection is evidently to the length of the element itself, as the length P of the perpendicular let fall from the focus on the tangent is to the length r of the focal vector of the point of contact; such, therefore, is, by (59), the ratio of $n^{-\frac{1}{2}}$ to 1, if the scalar n, in the *equation of the normal* (57); receive the value which corresponds to the centre of curvature of the meridian; therefore we have

$$\frac{\text{R}}{\text{N}} = \frac{\nu - \alpha}{r\epsilon - \alpha} = n = \frac{r^2}{\text{P}^2} = \frac{rr'}{\text{PP}'} = \frac{rr'}{pa}, \tag{61}$$

N denoting the length of the portion of the normal which is comprised between the meridian and the axis, and R denoting the length of the radius of curvature of the meridian. The projection of this radius on the focal vector is evidently the focal half chord of curvature, of which half chord the length may be here denoted by C; we see then that if we again project this half chord on the normal, the result is the normal itself, that is the portion N, because this double process of projection multiplies R twice successively by $n^{-\frac{1}{2}}$; and if, once more, the normal be projected on the focal vector, the third projection so obtained is equal in length to the semiparameter p, because, by (15) and (16),

$$\iota(r\epsilon - \alpha) + (r\epsilon - \alpha)\iota = 2p; \tag{62}$$

hence

$$\sqrt{(\mathrm{RN})} = \mathrm{C} = np = \frac{rr'}{a} = \frac{2rr'}{r + r'}, \tag{63}$$

that is, for any conic section, the geometrical mean between the radius of curvature and the normal is equal to the harmonic mean between the two focal distances; of which distances the second, namely r', is to be regarded as negative for the hyperbola, and infinite for the parabola, and the harmonic mean determined accordingly. We have also, for every conic section (if $r'a^{-1}$ be suitably interpreted),

$$\sqrt{n} = \frac{\mathrm{R}}{\mathrm{C}} = \frac{\mathrm{C}}{\mathrm{N}} = \frac{\mathrm{N}}{p} = \sqrt{\left(\frac{rr'}{pa}\right)}, \tag{64}$$

so that the semiparameter, the normal, the focal half chord of curvature, and the radius of curvature, are in continued geometrical progression: and the analysis may be verified, by calculating directly, on the same principles, the length of the normal, as follows:

$$\mathrm{N} = \sqrt{\{-(r\epsilon - \alpha)^2\}} = \sqrt{\{r^2(e^2 + 1) + 2r(p - r)\}} = \sqrt{\left(\frac{prr'}{a}\right)}. \tag{65}$$

The general relation of a conic section to a *directrix* is an immediate geometrical consequence of the equation (15), which has been here (in part) discussed, and may be regarded as its simplest interpretation. Some of the foregoing symbolical results respecting such a section admit of dynamical interpretation also; and, in particular, the expression $a\iota - a\epsilon$, which has been seen to represent, both in length and in direction, the perpendicular let fall from the second focus on the tangent, may suggest, by its composition, what is, however, a more immediate consequence of the equation (12), that in the undisturbed motion of a planet or comet about the sun, *the whole varying tangential velocity may be decomposed into two partial velocities, of which both are constant in magnitude, while one of them is constant in direction also.* The component velocity, which is constant in magnitude, but not in direction, is always in the plane of the orbit, and is perpendicular to the heliocentric radius vector of the body; the other component, which is constant in both magnitude and direction is parallel to the velocity at perihelion; and the magnitude of this fixed component is to the magnitude of the revolving one in the ratio of the excentricity e to unity. The author supposes that this theorem respecting a decomposition of the velocity in an excentric orbit is known, though he does not remember having met with it; but conceived that it might properly be mentioned here, as being a very easy and immediate consequence of the present analysis: respecting the general principles of which analysis, the reader is requested to consult the Abstract, already referred to, of the communication of November, 1844, printed at the commencement of the present volume of the Proceedings of the Academy. There is no difficulty in deducing, on the same principles, from formulæ of the present paper, the known differential equation,

$$\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = \mathrm{M}\left(\frac{2}{r} - \frac{1}{a} - \frac{p}{r^2}\right), \tag{66}$$

which connects the radial component of velocity with the heliocentric distance and the time, and may be integrated by the usual processes.