CALCULUS OF PRINCIPAL RELATIONS

By

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Calculus of Principal Relations. By Professor Sir W. R. Hamilton.

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The method of principal relations is an extension of that mode of analysis which Sir William Hamilton has applied before to the sciences of optics and dynamics; its nature and spirit may be understood from the following sketch.

Let $x_1, x_2, \ldots, x_n$ be any number n of functions of any one independent variable s, with which they are connected by any one given differential equation of the first order, but not of the first degree,

$$0 = f(s, x_1, \ldots x_n, ds, dx_1, \ldots dx_n), \tag{1}$$

and also by $n - 1$ other differential equations, of the second order, to which the calculus of variations conducts, as supplementary to the given equation (1), and which may be thus denoted:

$$\frac{f'(x_1) - df'(dx_1)}{f'(dx_1)} = \cdots = \frac{f'(x_n) - df'(dx_n)}{f'(dx_n)}; \tag{2}$$

Let, also, $a_1, \ldots a_n$ be the n initial values of the n functions $x_1, \ldots x_n$, and let $a'_1, \ldots a'_n$ be the initial values of their n derived functions or differential coefficients $x'_1 = \dfrac{dx_1}{ds}, \ldots x'_n = \dfrac{dx_n}{ds}$, corresponding to any assumed initial value a of the independent variable s. If we could integrate the system of the n differential equations (1) and (2), we should thereby obtain n expressions for the n functions $x_1, \ldots x_n$, of the forms

$$\left.\begin{array}{l} x_1 = \phi_1(s, a, a_1, \ldots a_n, a'_1, \ldots a'_n), \\ \cdots\cdots\cdots \\ x_n = \phi_n(s, a, a_1, \ldots a_n, a'_1, \ldots a'_n); \end{array}\right\} \tag{3}$$

and, by the help of the initial equation analogous to (1), might then eliminate $a'_1, \ldots a'_n$, and deduce a relation of the form

$$0 = \psi(s, x_1, \ldots x_n, a, a_1, \ldots a_n); \tag{4}$$

that is, a relation between the initial and final values of the $n + 1$ connected variables $s, x_1, \ldots, x_n$. Reciprocally, the author has found that if this one relation (4) were known, it would be possible thence to deduce expressions for the n sought integrals (3) of the system of the n differential equations (1) and (2), or for the n sought relations between $s, x_1, \ldots x_n$, and $a, a_1, \ldots a_n, a'_1, \ldots a'_n$, however large the number n may be; in such a manner that all these many relations (3) are implictly contained in the one relation (4), which latter relation the author proposes to call on this account the *principal integral relation*, or simply, the PRINCIPAL RELATION, of the problem.

For he has found that the n following equations hold good,

$$\frac{f'(ds)}{\psi'(s)} = \frac{f'(dx_1)}{\psi'(x_1)} = \cdots = \frac{f'(dx_n)}{\psi'(x_n)}; \tag{5}$$

which may be put under the forms

$$\left.\begin{aligned} a_1 &= \phi_1(a, s, x_1, \ldots x_n, x'_1, \ldots x'_n), \\ &\cdots\cdots\cdots \\ a_n &= \phi_n(a, s, x_1, \ldots x_n, x'_1, \ldots x'_n), \end{aligned}\right\} \tag{6}$$

and are evidently transformations of the n sought integrals (3). And with respect to the mode in which, without previously effecting the integrations (3), it is possible to determine the *principal relation* (4), or the *principal function* which it introduces, when it is conceived to be resolved, as follows, for the originally independent variable s,

$$s = \phi(x_1, \ldots x_n, a_1, \ldots a_n), \tag{7}$$

the author remarks that a partial differential equation of the first order may be assigned, which this principal function ϕ must satisfy, and also an initial condition adapted to remove the arbitrariness which otherwise would remain. In fact the equations (5) may be thus written,

$$\frac{\delta ds}{\delta dx_1} = \frac{\delta s}{\delta x_1}, \ldots \frac{\delta ds}{\delta dx_n} = \frac{\delta s}{\delta x_n}, \tag{8}$$

in which

$$\frac{\delta ds}{\delta dx_i} = -\frac{f'(dx_i)}{f'(ds)}, \quad \text{and} \quad \frac{\delta s}{\delta x_i} = \phi'(x_i), \tag{9}$$

and since, by (1), there subsists a known relation of the form

$$0 = F\left(s, x_1, \ldots x_n, \frac{\delta ds}{\delta dx_1}, \ldots \frac{\delta ds}{\delta dx_n}\right), \tag{10}$$

the following relation must also hold good,

$$0 = F\left(s, x_1, \ldots x_n, \frac{\delta s}{\delta x_1}, \ldots \frac{\delta s}{\delta x_n}\right), \tag{11}$$

that is, the principal function ϕ must satisfy the following partial differential equation of the first order,

$$0 = F(\phi, x_1, \ldots, x_n, \phi'(x_1), \ldots \phi'(x_n)); \tag{12}$$

it must also satisfy the following initial condition,

$$0 = \lim_{s=a} f(a, a_1, \ldots, a_n, \varphi - a, x_1 - a_1, \ldots x_n - a_n). \tag{13}$$

Such are the most essential principles of the new method in analysis which Sir William Hamilton has proposed to designate by the name of the *Method of Principal Relations*, and of which, perhaps, the simplest *type* is the formula

$$\frac{\delta ds}{\delta dx} = \frac{\delta s}{\delta x}, \tag{14}$$

to be interpreted like the equations (8).

The simplest *example* which can be given, to illustrate the meaning and application of these principles, is, perhaps, that in which the differential equations are

$$0 = \left(\frac{dx_1}{ds}\right)^2 + \left(\frac{dx_2}{ds}\right)^2 - 1, \tag*{(1)$'$}$$

and

$$\frac{ddx_1}{dx_1} = \frac{ddx_2}{dx_2}. \tag*{(2)$'$}$$

Here, ordinary integration gives

$$x_1 = a_1 + a'_1(s - a), \quad x_2 = a_2 + a'_2(s - a); \tag*{(3)$'$}$$

and consequently conducts to the following relation, (in this case the *principal* one,)

$$0 = (x_1 - a_1)^2 + (x_2 - a_2)^2 - (s - a)^2, \tag*{(4)$'$}$$

or

$$s = a + \sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2}, \tag*{(7)$'$}$$

because, by $(1)'$, we have

$$a'^2_1 + a'^2_2 = 1;$$

it enables us therefore to verify the relations (8) or (14), for it gives

$$\frac{\delta s}{\delta x_1} = \frac{x_1 - a_1}{s - a} = \frac{dx_1}{ds} = \frac{\delta ds}{\delta dx_1},$$

and in like manner,

$$\frac{\delta s}{\delta x_2} = \frac{\delta ds}{\delta dx_2}.$$

Reciprocally, in this example, the following known relation, deduced from $(1)'$,

$$0 = \left(\frac{\delta ds}{\delta dx_1}\right)^2 + \left(\frac{\delta ds}{\delta dx_2}\right)^2 - 1, \tag*{(10)$'$}$$

would have given, by the principles of the new method, this partial differential equation of the first order,

$$0 = \left(\frac{\delta s}{\delta x_1}\right)^2 + \left(\frac{\delta s}{\delta x_2}\right)^2 - 1, \tag*{(11)$'$}$$

which might have been used, in conjunction with the initial condition

$$0 = \lim_{s=a}. \left\{ \left(\frac{x_1 - a_1}{s - a} \right)^2 + \left(\frac{x_2 - a_2}{s - a} \right)^2 - 1 \right\}, \tag{13$'$}$$

to determine the form (7)$'$ of the principal function s; and thence might have been deduced, by the same new principles, the ordinary integrals (3)$'$, under the forms

$$a_1 = x_1 + a'_1(a - s), \quad a_2 = x_2 + a'_2(a - s). \tag{6$'$}$$

In so simple an instance as this, there would be no advantage in using the new method; but in a great variety of questions, including all those of mathematical optics, and mathematical dynamics, (at least, as those sciences have been treated by the author of this communication,) and in general all the problems in which it is required to integrate those systems of ordinary differential equations (whether of the second or of a higher order) to which the calculus of variations conducts, the method of principal relations assigns immediately a system of finite expressions for the integrals of the proposed equations, an object which can only very rarely be attained by any of the methods known before. It seems, for example, to be impossible, by any other method, to express rigorously, in finite terms, the integrals of the differential equations of motion of a system of many points, attracting or repelling one another; which yet was easily accomplished by a particular application of the general principles that have been here explained*. The author hopes to present these principles in a still more general form hereafter.

* See *Philosophical Transactions* for 1834 and 1835; also, *Report* of Edinburgh Meeting of the British Association.