# ON A PROOF OF PASCAL'S THEOREM BY MEANS OF QUATERNIONS; AND ON SOME OTHER CONNECTED SUBJECTS

**By**

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(Proceedings of the Royal Irish Academy, 3 (1847), pp. 273–292.)

Edited by David R. Wilkins

2000

*On a proof of Pascal's Theorem by means of Quaternions; and on some other connected Subjects.*

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Communicated July 20, 1846.

[*Proceedings of the Royal Irish Academy*, vol. 3 (1847), pp. 273–292.]

Sir William R. Hamilton read a paper on the expression and proof of Pascal's theorem by means of quaternions; and on some other connected subjects.

This proof of the theorem of Pascal depends on the following form of the general equation of cones of the second degree:

$$\mathrm{S} \,.\, \beta\beta'\beta'' = 0; \tag{1}$$

in which

$$\left.\begin{aligned} \beta &= \mathrm{V}(\mathrm{V} \,.\, \alpha\alpha' \,.\, \mathrm{V} \,.\, \alpha'''\alpha^{\mathrm{IV}}), \\ \beta' &= \mathrm{V}(\mathrm{V} \,.\, \alpha'\alpha'' \,.\, \mathrm{V} \,.\, \alpha^{\mathrm{IV}}\alpha^{\mathrm{V}}), \\ \beta'' &= \mathrm{V}(\mathrm{V} \,.\, \alpha''\alpha''' \,.\, \mathrm{V} \,.\, \alpha^{\mathrm{V}}\alpha), \end{aligned}\right\} \tag{2}$$

$\alpha$, $\alpha'$, $\alpha''$, $\alpha'''$, $\alpha^{\mathrm{IV}}$, $\alpha^{\mathrm{V}}$, being any six homoconic vectors, and S, V, being characteristics of the operations of taking separately the scalar and vector parts of a quaternion.

In all these geometrical applications of quaternions, it is to be remembered that *the product of two opposite vectors is a positive number*, namely the product of the numbers expressing the lengths of the two factors; and that the *product of two rectangular vectors is a third vector rectangular to both*, and such that the rotation round it, from the multiplier to the multiplicand, is positive. These conceptions, or definitions, of geometrical multiplication, are essential in the theory of quaternions, and are hitherto (so far as Sir William Hamilton knows) peculiar to it. If they be adopted, they oblige us to regard the *product* (or the quotient) *of two inclined vectors* (neither parallel nor perpendicular to each other), as being *partly a number and partly a line*; on which account a quaternion, generally, as being always, in its geometrical aspect, a product (or quotient) of two lines, may perhaps not improperly be also called a GRAMMARITHM (by a combination of the two Greek words γραμμή and ἀριθμός, which signify respectively a *line* and a *number*). In this phraseology, the scalar part of a quaternion would be the arithmic part of a grammarithm; and the vector part of a quaternion would be the grammic part of a grammarithm. In the form given above, of the general equation of cones of the second degree, the six symbols, $\alpha, \dots, \alpha^{\mathrm{V}}$ denote six edges of a hexahedral angle inscribed in such a cone; the six binary products $\alpha\alpha', \dots, \alpha^{\mathrm{V}}\alpha$, of those lines taken in their order, are grammarithms, of which the symbols $\mathrm{V} \,.\, \alpha\alpha'$, &c., denote the grammic parts, namely, certain lines perpendicular respectively to the six plane faces of the angle; the three products

$$\mathrm{V} \,.\, \alpha\alpha' \,.\, \mathrm{V} \,.\, \alpha'''\alpha^{\mathrm{IV}}, \quad \text{\&c.},$$

of normals to opposite faces, are again grammarithms, of which the grammic parts are the three lines $\beta$, $\beta'$, $\beta''$, situated respectively in the intersections of the three pairs of opposite faces of the angle inscribed in the cone; and the equation (1) of that cone, which expresses that the arithmic part of the product of these three lines vanishes, shows also, by the principles of this theory, that these lines themselves are *coplanar*; which is a form of the theorem of Pascal.

The rules of this calculus of grammarithms, or of quaternions, give, generally, for the arithmic or scalar part of the product of the vector parts of the three products of any six lines or vectors $\alpha\alpha'$, $\beta\beta'$, $\gamma\gamma'$, taken two by two, the following transformed expression:

$$\mathrm{S}(\mathrm{V}\,.\,\alpha\alpha'\,.\,\mathrm{V}\,.\,\beta\beta'\,.\,\mathrm{V}\,.\,\gamma\gamma') = \mathrm{S}\,.\,\alpha\gamma\gamma'\,.\,\mathrm{S}\,.\,\alpha'\beta\beta' - \mathrm{S}\,.\,\alpha'\gamma\gamma'\,.\,\mathrm{S}\,.\,\alpha\beta\beta'; \tag{3}$$

and by applying this general transformation to the recent results, we find easily, that the equation (1), under the conditions (2), may be put under the form:

$$\frac{\mathrm{S}\,.\,\alpha\alpha'\alpha''}{\mathrm{S}\,.\,\alpha\alpha'''\alpha''}\cdot\frac{\mathrm{S}\,.\,\alpha''\alpha'''\alpha^{\mathrm{IV}}}{\mathrm{S}\,.\,\alpha''\alpha'\alpha^{\mathrm{IV}}} = \frac{\mathrm{S}\,.\,\alpha\alpha'\alpha^{\mathrm{V}}}{\mathrm{S}\,.\,\alpha\alpha'''\alpha^{\mathrm{V}}}\cdot\frac{\mathrm{S}\,.\,\alpha^{\mathrm{V}}\alpha'''\alpha^{\mathrm{IV}}}{\mathrm{S}\,.\,\alpha^{\mathrm{V}}\alpha'\alpha^{\mathrm{IV}}}; \tag{4}$$

which is another mode of expressing by quaternions the general condition required, in order that six vectors $\alpha, \ldots, \alpha^{\mathrm{V}}$, diverging from one common origin, may all be sides of one common cone of the second degree. The summit of this cone, or the common initial point of each of these six vectors, being called O, let the six final points be A B C D E C$'$: the transformed *equation of homoconicism* (4), expresses that the *ratio compounded of the two ratios of the two pyramids* OABC, OCDE, *to the two other pyramids* OADC, OCBE, *does not change when we pass from the point* C *to any other point* C$'$ *on the same cone of the second degree*: which is a form of the theorem of M. Chasles, respecting the constancy of the anharmonic ratio. An intimate connexion between this theorem and that of Pascal is thus exhibited, by this symbolical process of transformation.

As the equation (1) expresses that the *three* vectors $\beta\,\beta'\,\beta''$ are *coplanar*, or that they are contained on one common plane, if they diverge from one common origin, and as the equation (4) expresses that the *six* vectors $\alpha, \ldots, \alpha^{\mathrm{V}}$ are *homoconic*, so does this other equation,

$$\mathrm{S}\,.\,\rho(\rho-\gamma)(\gamma-\beta)(\beta-\alpha)\alpha = 0, \tag{5}$$

express that the *four* vectors $\alpha$, $\beta$, $\gamma$, $\rho$ are *homosphæric*, or that they may be regarded as representing, in length and in direction, *four diverging chords of one common sphere.* Thus, the *arithmic part of the continued product of the five successive sides of any rectilinear* (but not necessarily plane) *pentagon, inscribed in a sphere, is zero*; and conversely, if in any investigation respecting any rectilinear, but, generally, uneven, pentagon ABCDE in space, the product AB $\times$ BC $\times$ CD $\times$ DE $\times$ EA of five successive sides, when determined by the rules of the present calculus, is found to be a pure vector, or can be entirely constructed by a *line*, so that in a notation already submitted to the Academy (see account of the communication made in last December) the equation

$$\mathrm{S}\,.\,\mathrm{ABCDEA} = 0, \tag{6}$$

is found to be satisfied, we may then infer that the five corners, A, B, C, D, E, of this pentagon, are situated on the surface of one common sphere. This *equation of homosphæricism* (5) or (6), appears to the present author to be very fertile in its consequences. To leave no doubt respecting its *meaning*, and to present it under a form under which it may be easily understood by those who have not yet made themselves masters of the whole of the theory, it may be stated thus: if we write for abridgment,

$$\left.\begin{aligned}
\alpha_1 &= i(x_1 - x_2) + j(y_1 - y_2) + k(z_1 - z_2), \\
\alpha_2 &= i(x_2 - x_3) + j(y_2 - y_3) + k(z_2 - z_3), \\
\alpha_3 &= i(x_3 - x_4) + j(y_3 - y_4) + k(z_3 - z_4), \\
\alpha_4 &= i(x_4 - x_5) + j(y_4 - y_5) + k(z_4 - z_5), \\
\alpha_5 &= i(x_5 - x_1) + j(y_5 - y_1) + k(z_5 - z_1),
\end{aligned}\right\} \tag{7}$$

and then develop the continued product of these five expressions, *using the distributive, but not* (so far as relates to $i\ j\ k$) *the commutative property of multiplication*, and reducing the result to the form of a quaternion,

$$\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5 = w + ix + jy + kz, \tag{8}$$

by the fundamental symbolical relations between the *three coordinate characteristics* $i\ j\ k$, which were communicated to the Academy by Sir William Hamilton in November, 1843, and which may be thus concisely stated:*

$$i^2 = j^2 = k^2 = ijk = -1; \tag{A}$$

and if we find, as the result of this calculation, that the term $w$, or the part of the quaternion (8) which is independent of the characteristics $i\,j\,k$, vanishes, so that we have the following equation, which is entirely freed from those symbolic factors,

$$w = 0, \tag{9}$$

we shall then know that the points, of which the rectangular coordinates are respectively $(x_1\,y_1\,z_1)$ $(x_2\,y_2\,z_2)$ $(x_3\,y_3\,z_3)$ $(x_4\,y_4\,z_4)$ $(x_5\,y_5\,z_5)$, are *five homosphæric points*, or that one common spheric surface will contain them all.

The actual process of this multiplication and reduction would be tedious, nor is it offered as the easiest, but only as *one* way of forming the equation in rectangular coordinates, which is here denoted by (9). A much easier way would be to prepare the equation (5) by a previous development, so as to put it under the following form:

$$\rho^2\,\mathrm{S}\,.\,\alpha\beta\gamma = \alpha^2\,\mathrm{S}\,.\,\beta\gamma\rho + \beta^2\,\mathrm{S}\,.\,\gamma\alpha\rho + \gamma^2\,\mathrm{S}\,.\,\alpha\beta\rho; \tag{10}$$

* These fundamental equations between the author's symbols $i$, $j$, $k$, appeared, under a slightly more developed form, in the number of the *London, Edinburgh and Dublin Philosophical Magazine* for July, 1844; in which Magazine the author has continued to publish, from time to time, some articles of a Paper on Quaternions; reserving, however, for the Transactions of the Royal Irish Academy, a more complete and systematic account of his researches on this extensive subject.

which also admits of a simple geometrical interpretation. For, by comparing it with the following equation, which is in this calculus an *identical* one, or is satisfied by *any* four vectors, $\alpha$, $\beta$, $\gamma$, $\rho$:

$$\rho\,\mathrm{S}\,.\,\alpha\beta\gamma = \alpha\,\mathrm{S}\,.\,\beta\gamma\rho + \beta\,\mathrm{S}\,.\,\gamma\alpha\rho + \gamma\,\mathrm{S}\,.\,\alpha\beta\rho, \tag{11}$$

we find that the form (10) gives

$$\rho^2 = \alpha\alpha' + \beta\beta' + \gamma\gamma', \tag{12}$$

if $\alpha'$, $\beta'$, $\gamma'$ denote three diverging edges of a parallelepiped, of which the intermediate diagonal (or their symbolic sum) is the chord $\rho$ of a sphere, while $\alpha\,\beta\,\gamma$ are three other chords of the same sphere, in the directions of the three edges, and coinitial with them and with $\rho$; so that *the square on the diagonal $\rho$ is equal to the sum of the three rectangles under the three edges $\alpha'\,\beta'\,\gamma'$ and the three chords $\alpha\,\beta\,\gamma$*, with which, in direction, those edges respectively coincide. This theorem is only mentioned here, as a simple example of the *interpretation* of the formulae to which the present method conducts; since the same result may be obtained very simply from a more ordinary form of the equation of the sphere, referred to the edges $\alpha'\,\beta'\,\gamma'$ as oblique coordinates; and doubtless, has been already obtained in that or in some other way. An analogous theorem for the ellipsoid may be obtained with little difficulty.

If we suppose in the formula (6), that the point E of the pentagon approaches to the point A, the side EA tends to become an infinitely small tangent to the sphere; and thus we find that V . ABCDA, or that *the vector part of the continued product* AB × BC × CD × DA*, of the four sides of an uneven* (or *gauche*) *quadrilateral* ABCD, if determined by the rules of multiplication proper to this calculus, *is normal to the circumscribed sphere* at the point A, where the first and fourth sides are supposed to meet. By the non-commutative character of quaternion multiplication, we should get a different product, if we took the factors in the order BC × CD × DA × AB; and accordingly the vector or grammic part V . BCDAB of this new quaternion product would represent a new line in space, namely, a normal to the same sphere at B: and similarly may the normals be found at the two other corners of the quadrilateral, by two other arrangements of the four sides as factors. To determine the lengths of the normal lines thus assigned, we may observe that if A′ B′, C′, D′ be the four points on the same sphere, which are diametrically opposite to the four given points A B, C, D, then the four diameters A′A, B′B, C′C, D′D, are given by four expressions, of which it may be sufficient to write one, namely

$$\mathrm{A'A} = \frac{\mathrm{V\,.\,ABCDA}}{\mathrm{S\,.\,ABCD}}. \tag{13}$$

The denominator of this expression denotes (as was remarked in a former communication) the sextuple volume of the pyramid, or tetrahedron, ABCD; it vanishes, therefore, when the four points A, B, C, D are in one plane: so that we have for *any plane quadrilateral* the equation,

$$\mathrm{S\,.\,ABCD} = 0. \tag{14}$$

If the sphere is then to become only *indeterminate*, and not necessarily infinite, we must suppose that the numerator of the same expression (13) also vanishes, that is, we must have in this case the condition

$$\mathrm{V\,.\,ABCDA} = 0. \tag{15}$$

In words, as the product of the five successive sides of an uneven but rectilinear pentagon inscribed in a sphere, has been seen to be *purely a line*, so we now see that *the product of the four successive sides of a quadrilateral inscribed in a circle is* (in this system) *purely a number*: whereas, for *every other* rectilinear quadrilateral, *whether plane or gauche*, the grammarithm obtained as the product of four successive sides *involves a grammic part*, which does not vanish. This condition (15), for a quadrilateral inscribable in a circle, could not be always satisfied, when D approached to A, and tended to coincide with it, unless the following theorem were also true, which can accordingly be otherwise proved: *the product* ABCA, or AB × BC × CA, *of three successive sides of any triangle* ABC*, is a pure vector, in the direction of the tangent to the circumscribed circle*, at the point A, where the sides which are assumed as first and third factors of the product meet each other. If $\mathrm{A}_{\prime}$ be the point upon this circumscribed circle which is diametrically opposite to A, we find for the length and direction of the diameter $\mathrm{AA}_{\prime}$ in this notation, that is, for the straight line *to* A *from* $\mathrm{A}_{\prime}$, the expression:*

$$\mathrm{AA}_{\prime} = \frac{\mathrm{ABCA}}{\mathrm{V}\,.\,\mathrm{ABC}}; \tag{16}$$

the denominator denoting a line which is in direction perpendicular to the plane of the triangle, and in magnitude represents the double of its area; while the numerator is, as we have just seen, in direction tangential to the circle at A, and its length represents the product of the lengths of the three sides, or the volume of the solid constructed with those sides as rectangular edges. We may add, that this tangential line ABCA is distinguished from the equally long but *opposite* tangent ACBA to the same circle ABC at the same point A, by the condition that the former is intermediate in direction, between AB (prolonged through A) and CA, while the latter in like manner lies between AC (prolonged) and BA: or we may say that the line ABCA touches, at A, the segment *alternate* to that segment of the circle ABC which has for AC the base, and contains the point B; while the opposite line ABCA touches, at the same point, the last mentioned segment itself. The condition for the diameter $\mathrm{AA}_{\prime}$ becoming infinite, or for the three points ABC being situated on one common straight line, is

$$\mathrm{V}\,.\,\mathrm{ABC} = 0. \tag{17}$$

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* With respect to the *notation of division*, in this theory, the author proposes to distinguish between the two symbols

$$\mathrm{Q}^{-1}\mathrm{Q}' \quad \text{and} \quad \frac{\mathrm{Q}}{\mathrm{Q}'},$$

which he inadvertently used as interchangeable in his first communication to the Academy: and to make them satisfy the two separate equations,

$$\mathrm{Q} \times \mathrm{Q}^{-1}\mathrm{Q}' = \mathrm{Q}';$$

$$\frac{\mathrm{Q}'}{\mathrm{Q}} \times \mathrm{Q} = \mathrm{Q}'.$$

He proposes to confine the symbol $\mathrm{Q}' \div \mathrm{Q}$ to the signification thus assigned for the latter of the two symbols which have been thus defined, and which, on account of the non-commutative property of multiplication of quaternions, ought not to be confounded with each other.

This formula (17) is therefore, in this notation, the *general equation of a straight line* in space; (15) is the general *equation of a circle*; (14) of a plane, and (6) of a sphere.* It may seem strange that the line and circle should here be represented each by only *one* equation; but these equations are of *vector forms*, and decompose themselves each into three equations, equivalent, however, only to two distinct ones, when we pass to rectangular coordinates, for the sake of comparison with known results.

In the same notation of capitals, whatever five distinct points may be denoted by A, B, C, D, E, we have the general transformation,

$$\mathrm{ABCDEA} = \mathrm{ABCA} \times \mathrm{ACDA} \times \mathrm{ADEA} \div \mathrm{ACADA}, \tag{18}$$

in which the divisor ACADA, or ACA × ADA, is the product of two positive scalars; if then we had otherwise established the interpretation lately assigned to the symbol ABCA, as denoting a line which touches at A the circle ABC, we might have in that way deduced the equation (6) of a sphere, as the *condition of coplanarity of the three tangents* at A, *to the three circles*, ABC, ACD, ADE. And we see that when this condition is satisfied, so that the points A, B, C, D, E are homosphæric, and that, therefore, the symbol ABCDEA represents a vector, we can *construct the direction of this vector* by drawing in the plane which touches the sphere at A, a line $\mathrm{A}_1\mathrm{A}_2$ parallel to the line ACDA which touches the circle ACD at A, and cutting, in the points $\mathrm{A}_1$ and $\mathrm{A}_2$, the two lines ABCA and ADEA, which are drawn at A to touch the circles ABC, ADE; for then the vector ABCDEA, which is thus seen to be a tangent to the sphere, will touch, at the same point A, the circle $\mathrm{AA}_1\mathrm{A}_2$, described on the tangent plane. In the more general case, when the condition (6) is *not* satisfied, and when, therefore, the rectilinear pentagon ABCDE, which we shall suppose to be uneven, cannot be inscribed in a sphere, the scalar symbol S . ABCDEA which has been seen to vanish, when the pentagon can be so inscribed, represents the continued *product of the lengths of the five sides* AB, BC, CD, DE, EA, *multiplied by the sextuple volume of that triangular pyramid which is constructed with three coterminous edges, each equal to the unit of length, and touching at the vertex* A *the three circles* ABC, ACD, ADE, *which have respectively for chords the three remote sides of the pentagon*, and are not now homosphæric circles. And because, in general, in this notation, the equation

$$\mathrm{S} \,.\, \mathrm{ABCDEA} = \mathrm{S} \,.\, \mathrm{BCDEAB} \tag{19}$$

holds good, it follows that *for any rectilinear pentagon* (in space) *the five triangular pyramids constructed on the foregoing plan*, with the five corners of the pentagon for their respective vertices, *have equal volumes.*

Besides the characteristics S and V, which serve to decompose a quaternion Q into *two parts*, of distinct and determined kinds, the author frequently finds it to be convenient to use two other characteristics of operation, T and U, which serve to decompose the same quaternion into *two factors*, of kinds equally distinct and equally determinate; in such a manner that we may write generally, with these characteristics, for any quaternion Q,

$$\mathrm{Q} = \mathrm{SQ} + \mathrm{VQ} = \mathrm{TQ} \times \mathrm{UQ}. \tag{20}$$

* The simpler equation of scalar form, S . ABC = 0, also represents a spheric surface, if B be regarded as the variable point; but a plane, if B be fixed, and either A or C alone variable.

The factor $\mathrm{TQ}$ is always a positive, or rather an absolute (or *signless*) number; it is what was called by the author, in his first communication on this subject to the Academy, the *modulus*, but he has since come to prefer to call it the TENSOR of the quaternion $\mathrm{Q}$: and he calls the other factor $\mathrm{UQ}$ the VERSOR of the same quaternion. As the *scalar of a sum is the sum of the scalars*, and the *vector of a sum is the sum of the vectors*, so the *tensor of a product is the product of the tensors*, and the *versor of a product is the product of the versors*; relations or properties which may be concisely expressed by the formulae:

$$\mathrm{S}\Sigma = \Sigma\mathrm{S}; \quad \mathrm{V}\Sigma = \Sigma\mathrm{V}; \tag{21}$$

$$\mathrm{T}\Pi = \Pi\mathrm{T}; \quad \mathrm{U}\Pi = \Pi\mathrm{U}. \tag{22}$$

When we operate by the characteristics $\mathrm{T}$ and $\mathrm{U}$ on a straight line, regarded as a vector, we obtain as the *tensor* of this line a *signless number* expressing its *length*; and, as the *versor* of the same line, an *imaginary unit*, determining its *direction.* When we operate on the product $\mathrm{ABC} = \mathrm{AB} \times \mathrm{BC}$ of two successive lines, regarded as a quaternion, we obtain for the tensor, $\mathrm{T} \,.\, \mathrm{ABC}$, the product of the lengths of the two lines, or the area of the rectangle under them; and for the versor of the same product of two successive sides of a triangle (or polygon), we obtain an expression of the form

$$\mathrm{U} \,.\, \mathrm{ABC} = \cos \mathrm{B} + \sqrt{-1} \sin \mathrm{B}; \tag{23}$$

the symbol $\mathrm{B}$ in the second member denoting the *internal angle* of the figure at the point denoted by the same letter, which angle is thus the *amplitude of the versor*, and at the same time (in the sense of the author's first communication) the *amplitude of the quaternion* itself, which quaternion is here denoted by the symbol $\mathrm{ABC}$. In this theory (as was shown by the author to the Academy in that first communication), there are *infinitely many* different square roots of negative unity, constructed by lines equal to each other, and to the unit of length, but distinguishable by their *directional* (or *polar*) coordinates: the *particular* $\sqrt{-1}$ which enters into the expression (23) is perpendicular to the plane of the triangle $\mathrm{ABC}$. It is the *versor of the vector* of that quaternion which is denoted by the same symbol $\mathrm{ABC}$; and it may, therefore, be replaced by the symbol $\mathrm{UV} \,.\, \mathrm{ABC}$, which we may agree to abridge to $\mathrm{W} \,.\, \mathrm{ABC}$, so that we may establish the symbolic equation:

$$\mathrm{UVQ} = \mathrm{WQ}, \quad \text{or simply,} \quad \mathrm{UV} = \mathrm{W}; \tag{24}$$

we may also call $\mathrm{WQ}$ the *vector unit* of the quaternion $\mathrm{Q}$. The expression (23) suggests also the denoting the *amplitude* of *any* quaternion by the geometrical mark for an *angle*, which notation will also agree with the original conception of such an amplitude; and thus we are led to write, generally, as a transformed expression for a versor,

$$\mathrm{UQ} = \cos \angle\mathrm{Q} + \mathrm{WQ} \,.\, \sin \angle\mathrm{Q}. \tag{25}$$

The amplitude of a vector is in this theory a quadrant; that of a positive number being, as usual, zero, and that of a negative number two right angles. Applying the same principles and notation to the case of the continued product $\mathrm{ABCDA}$ of the four successive sides of an uneven quadrilateral ABCD, we find that the amplitude $\angle$ABCDA of this quaternion product is equal to the *angle of the lunule* ABCDA, if we employ this term "lunule" to denote a portion of a spherical surface bounded by two arcs (which may be greater than halves) of *small* circles, namely, here, the portion of the surface of the sphere circumscribed about the quadrilateral ABCD, which portion is bounded by the two arcs that go from the corner A of that quadrilateral to the opposite corner C, and which pass respectively through the two other corners B and D. The tensor and scalar of the continued product of the four sides of the quadrilateral do not change when the sides are taken in the order, second, third, fourth, first; and generally,

$$\cos \angle \mathrm{Q} = \mathrm{SQ} \div \mathrm{TQ}; \tag{26}$$

so that we have the equation

$$\angle \mathrm{ABCDA} = \angle \mathrm{BCDAB}; \tag{27}$$

hence the *two lunules* ABCDA and BCDAB, which have for their *diagonals* AC and BD the two diagonals of the quadrilateral, and with which the lunules CDABC and DABCD respectively coincide, are *mutually equiangular* at A and B. Thus, generally, for *any four points*, A B C D, the two circles ABC, ADC cross each other at A and C (in space, or on one plane), under the same angles as the two other circles, BCD, BAD, at B and D.

Again, it may be remarked, that the condition for a fifth point E being contained on the plane which touches, at A, the sphere circumscribed about the tetrahedron ABCD, is expressed by the equation

$$\mathrm{S} \,.\, \mathrm{ABCDAE} = 0; \tag{28}$$

this equation, therefore, ought not to be compatible with the equation (6), which expressed that the point E was on the sphere itself, except by supposing that the point E coincides with the point of contact A; and accordingly the principles and rules of this notation give, generally,

$$\mathrm{S} \,.\, \mathrm{ABCDEA} + \mathrm{S} \,.\, \mathrm{ABCDAE} = \mathrm{S} \,.\, \mathrm{ABCD} \,.\, \mathrm{AEA}, \tag{29}$$

in which, by (14) the first factor S . ABCD of the second member does not vanish if the sphere be finite, that is, if the volume of the tetrahedron does not vanish, while the second factor may be thus transformed,

$$\mathrm{AEA} = -(\mathrm{EA})^2, \tag{30}$$

so that the coexistence of the two equations (6) and (28) of the sphere and its tangent plane, is thus seen to require that we shall have

$$\mathrm{EA} = 0; \tag{31}$$

which is, relatively to the sought position of E, the *equation of the point of contact.* These examples, though not the most important that might be selected, may suffice to show that there already exists a *calculus*, which may deserve to be further developed, for combining and transforming geometrical expressions of this sort. Several of the elements of such a calculus, especially as regards geometrical *addition* and *subtraction*, have been contributed by other, and (as the author willingly believes) by better geometers; what Sir William Hamilton considers to be peculiarly his own contribution to this department of mathematical and symbolical science consists of the introduction and development of those conceptions of GEOMETRICAL MULTIPLICATION (and division), which were embodied by him (in 1843) in his fundamental formulae for the *symbolic squares and products of the three coordinate characteristics* (or algebraically imaginary units) $i$, $j$, $k$, which entered into his original expression of a QUATERNION $(w + ix + jy + kz)$, and by which he succeeded in representing, symmetrically, that is, *without any selection of one direction as eminent*, the three dimensions of space.

It is, however, convenient, in many researches, to retain the notation in which Greek letters denote *vectors*, instead of employing that other notation, in which capital letters (a few *characteristics* excepted), denote *points*. In the former notation it was shown to the Academy in last December (see formula (21) of the abstract of the author's communication of that date), that the equation of an ellipsoid, with three unequal axes, referred to its centre as the origin of vectors, may be put under the form:*

$$(\alpha\rho + \rho\alpha)^2 - (\beta\rho - \rho\beta)^2 = 1;$$

$\rho$ being the variable vector of the ellipsoid, and $\beta$ and $\alpha$ being two constant vectors, in the directions respectively of the axes of one of the two circumscribed cylinders of revolution, and of a normal to the plane of the corresponding ellipse of contact. Decomposing the first member of that equation of an ellipsoid into two factors of the first degree, or writing the equation as follows:

$$(\alpha\rho + \rho\alpha + \beta\rho - \rho\beta)(\alpha\rho + \rho\alpha - \beta\rho + \rho\beta) = 1, \tag{32}$$

we may observe that these two factors, which are thus *separately linear* with respect to the variable vector $\rho$, are at the same time *conjugate quaternions*; if we call two quaternions, $\mathrm{Q}$ and $\mathrm{KQ}$, CONJUGATE, when they have *equal scalars* but have *opposite vectors*, so that generally,

$$\mathrm{KQ} = \mathrm{SQ} - \mathrm{VQ}, \quad \text{or, more concisely,} \quad \mathrm{K} = \mathrm{S} - \mathrm{V}. \tag{33}$$

And if we further observe, that in general the *product of two conjugate quaternions* is equal to *the square of their common tensor*,

$$\mathrm{Q} \times \mathrm{KQ} = (\mathrm{SQ})^2 - (\mathrm{VQ})^2 = (\mathrm{TQ})^2, \tag{34}$$

we shall perceive that the equation (32) of an ellipsoid may be put, by extraction of a square root, under this simpler, but not less general form:

$$\mathrm{T}(\alpha\rho + \rho\alpha + \beta\rho - \rho\beta) = 1. \tag{35}$$

Again, by employing the principle, that $\mathrm{T}\Pi = \Pi\mathrm{T}$, we may again decompose the first member of (35) into two factors, and may write the *equation of an ellipsoid* thus:

$$\mathrm{T}(\alpha + \beta + \sigma) \,.\, \mathrm{T}\rho = 1, \tag{36}$$

* Appendix, No. V., page lviii. [*Proceedings of the Royal Irish Academy*, vol. 3.]

if we introduce an auxiliary vector, $\sigma$, connected with the vector $\rho$ by the relation

$$\sigma = \rho(\alpha - \beta)\rho^{-1}, \tag{37}$$

which gives, by the same principle respecting the tensor of a product,

$$\mathrm{T}\sigma = \mathrm{T}(\alpha - \beta); \tag{38}$$

so that the auxiliary vector $\sigma$ has a *constant length*, although it has by (37) a *variable direction*, depending on, and in its turn assisting to determine or construct the direction of the vector $\rho$ of the ellipsoid; for the same equation (37) gives for the *versor* of that vector the expression

$$\mathrm{U}\rho = \pm\mathrm{U}(\alpha - \beta + \sigma). \tag{39}$$

Hence, by the second general decomposition (20), and by the equation (36), the last-mentioned vector $\rho$ may be expressed as follows:

$$\rho = \frac{\mathrm{U}(\alpha - \beta + \sigma)}{\mathrm{T}(\alpha + \beta + \sigma)}; \tag{40}$$

making then, in the notation of capital letters for points,

$$\alpha + \beta = \mathrm{CB}, \quad \alpha - \beta = \mathrm{CA}, \quad \sigma = \mathrm{DC}, \quad \rho = \mathrm{EA}, \tag{41}$$

so that A is the centre of the ellipsoid, E a variable point on its surface, C the fixed centre of an auxiliary sphere, of which the surface passes through the fixed point A, and also through the auxiliary and variable point D, while B is another fixed point, we obtain the equation:

$$\mathrm{EA} = \pm\mathrm{U} \,.\, \mathrm{DA} \div \mathrm{T} \,.\, \mathrm{DB}; \tag{42}$$

which gives

$$(\mathrm{EA})^{-1} = \mp\mathrm{U} \,.\, \mathrm{DA} \,.\, \mathrm{T} \,.\, \mathrm{DB}, \tag{43}$$

and shows, therefore, that the *proximity* $(\mathrm{EA})^{-1}$ *of a variable point* E*, on the surface of an ellipsoid, to the centre* A *of that ellipsoid, is represented in direction by a variable chord* DA *of a fixed sphere*, of which one extremity A is fixed, while the *magnitude* of the same proximity, or the *degree of nearness* (increasing as E approaches to the centre A, and diminishing as it recedes), is *represented by the distance* DB *of the other extremity* D *of the same chord* DA *from another fixed point* B which may be supposed to be external to the sphere. This use of the *word* "proximity," which appears to be a very convenient one, is borrowed from Sir John Herschel: the *construction for the ellipsoid* is perhaps new, and may be also thus enunciated:—From a fixed point A on the surface of a sphere, draw a variable chord DA; let D′ be the second point of intersection of the spheric surface with the secant DB, drawn to the variable extremity D of this chord from a fixed external point B; take the radius vector EA equal in length to D′B, and in direction either coincident with, or opposite to, the chord DA; *the locus of the point* E*, thus constructed, will be an ellipsoid*, which will pass through the point B. This fixed point B (one of four known points upon the principal ellipse) may, perhaps, be fitly called a POLE, and the line BE a *polar chord*, of the ellipsoid; and in the construction just stated, the two variable points D, D′ may be said to be *conjugate guide-points*, at the extremities of coinitial and *conjugate guide-chords* DA, D′A of a fixed *guide-sphere*, which passes through the centre A of the ellipsoid.

We may also say, that *if of a quadrilateral* (ABED′) *of which one side* (AB) *is given in length and in position, the two diagonals* (AE, BD′) *be equal to each other in length, and intersect* (in D) *on the surface of a given sphere* (with centre C), *of which a chord* (AD′) *is a side of the quadrilateral adjacent to the given side* (AB), *then the other side* (BE), *adjacent to the same given side, is a (polar) chord of a given ellipsoid*: of which last surface, the form, position, and magnitude, are thus seen to depend on the form, position, and magnitude, of what may, therefore, be called the *generating triangle* ABC. Two sides of this triangle, namely, BC and CA, are perpendicular to the two planes of *circular section*; and the third side AB is perpendicular to one of the two planes of *circular projection* of the ellipsoid, being the axis of revolution of a circumscribed circular cylinder. Many fundamental properties of the ellipsoid may be deduced with extreme facility, as geometrical* consequences of this mode of generation; for example, the well-known proportionality of the difference of the squares of the reciprocals of the semi-axes of a diametral section to the product of the sines of the inclinations of its plane to the two planes of circular section, presents itself under the form of a proportionality of the same difference of squares to the rectangle under the projections of the two sides BC and CA of the generating triangle on the plane of the elliptic section.

If we put the equation (35) of an ellipsoid under the form

$$\mathrm{T}(\iota\rho + \rho\kappa) = \kappa^2 - \iota^2, \tag{44}$$

the constant vectors $\iota$ and $\kappa$ will be in the directions of the normals to the planes of circular section, and may represent the two sides BC and AC of the triangle, while $\iota - \kappa$ will be one value of the variable vector $\rho$ or EA, namely, the remaining side of the same triangle, or the semi-diameter BA in the last-mentioned construction of the surface; and by applying to this equation (44) the general methods which the author has established for investigating by quaternions the tangent planes and curvatures of surfaces, it is found that the *vector of proximity* $\nu$ of the tangent plane to the centre of the ellipsoid (that is, the reciprocal of the perpendicular let fall on this plane from this centre), is determined in length and in direction by the equation,

$$(\kappa^2 - \iota^2)^2\nu = (\kappa^2 + \iota^2)\rho + \iota\rho\kappa + \kappa\rho\iota; \tag{45}$$

while the two rectangular directions of a vector $\tau$, tangential to a line of curvature, at the extremity of the vector $\rho$, are determined by the system of equations:

$$\nu\tau + \tau\nu = 0; \quad \nu\tau\iota\tau\kappa - \kappa\tau\iota\tau\nu = 0; \tag{46}$$

which may be thus written:

$$\mathrm{S}\,.\,\nu\tau = 0; \quad \mathrm{S}\,.\,\nu\tau\iota\tau\kappa = 0. \tag{47}$$

* For the following geometrical corollary, from the construction assigned above, the author is indebted to the Rev. J. W. Stubbs, Fellow of Trinity College. If the auxiliary point D describe, on the sphere, a circle of which the plane is perpendicular to BC, the point E on the ellipsoid will describe a spherical conic.

Of these two equations (46) or (47), the former expresses merely that the tangential vector $\tau$ is perpendicular to the normal vector $\nu$; while the latter is found to express that the tangent to either line of curvature of an ellipsoid is equally inclined to the two traces of the planes of circular section on the tangent plane, and therefore bisects one pair of the angles formed by the two circular sections themselves, which pass through the given point of contact. Indeed, it is easy to prove this relation of bisection otherwise, not only for the ellipsoid, but for the hyperboloids, by considering the common sphere which contains the circular sections last mentioned; the author believes that the result has been given in one of the excellent geometrical works of M. Chasles; it may also be derived without difficulty from principles stated in the masterly Memoir on Surfaces of the Second Order, which has been published by Professor Mac Cullagh in the Proceedings of this Academy. (See Part VIII, page 484.)

The length to which the present abstract has already extended, prevents Sir William Hamilton from offering on the present occasion any details respecting the processes (analogous in some respects to the calculi of variations and partial differentials) by which he applies the principles of his own method to investigations respecting surfaces and curves in space, or to physical problems connected therewith; he desires, however, to mention here that, in investigations respecting normals to surfaces, he finds it convenient to employ a new characteristic of operation of the form

$$(\mathrm{S} \,.\, \mathrm{d}\rho)^{-1} \,.\, \mathrm{d} = \sqsubset\!\!\!\!\!\!\sqsubset , \tag{48}$$

in order to obtain from a scalar function of a variable vector $\nu$ which shall be normal to the locus for which that scalar function is constant; and that the following more general characteristic of operation,

$$i\frac{\mathrm{d}}{\mathrm{d}x} + j\frac{\mathrm{d}}{\mathrm{d}y} + k\frac{\mathrm{d}}{\mathrm{d}z} = \triangleleft, \tag{49}$$

in which $x$, $y$, $z$ are ordinary rectangular coordinates, while $i$, $j$, $k$ are his own coordinate imaginary units, appears to him to be one of great importance in many researches. This will be felt (he thinks) as soon as it is perceived that with this meaning of $\triangleleft$ the equation

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 + \left(\frac{\mathrm{d}}{\mathrm{d}y}\right)^2 + \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^2 = -\triangleleft^2, \tag{50}$$

is satisfied in virtue of the fundamental relations between the symbols $i$, $j$, $k$; which relations give also, as another result of operating with the same characteristic, this other important symbolic expression, which presents itself under the form of a quaternion:

$$\triangleleft(it+ju+kv) = -\left(\frac{\mathrm{d}t}{\mathrm{d}x} + \frac{\mathrm{d}u}{\mathrm{d}y} + \frac{\mathrm{d}v}{\mathrm{d}z}\right) + i\left(\frac{\mathrm{d}v}{\mathrm{d}y} - \frac{\mathrm{d}u}{\mathrm{d}z}\right) + j\left(\frac{\mathrm{d}t}{\mathrm{d}z} - \frac{\mathrm{d}v}{\mathrm{d}x}\right) + k\left(\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{\mathrm{d}t}{\mathrm{d}y}\right). \tag{51}$$