ON THE CONSTRUCTION OF THE ELLIPSOID BY TWO SLIDING SPHERES

By

William Rowan Hamilton

(Proceedings of the Royal Irish Academy, 4 (1850), p. 341–342.)

Edited by David R. Wilkins

2000

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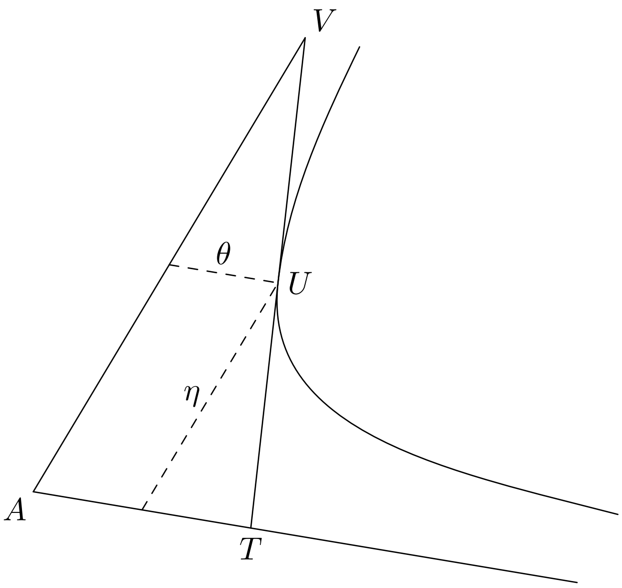
By Sir William R. Hamilton.

Communicated April 23, 1849.

[*Proceedings of the Royal Irish Academy*, vol. 4 (1850), p. 341–342.]

The following extract of a letter from Sir William Rowan Hamilton to the Rev. Charles Graves was read to the Academy:



"If I had been more at leisure when last writing, I should have remarked that besides the construction of the ellipsoid by the two *sliding spheres*, which, in fact, led me last summer to an equation nearly the same as that lately submitted to the Academy, a simple interpretation may be given to the equation,

$$\mathrm{TV}\frac{\eta\rho - \rho\theta}{\mathrm{U}(\eta - \theta)} = \theta^2 - \eta^2; \tag{1}$$

which may also be thus written,

$$\mathrm{TV}\frac{\rho\eta - \theta\rho}{\eta - \theta} = \frac{\theta^2 - \eta^2}{\mathrm{T}(\eta - \theta)}. \tag{2}$$

"At an umbilic U, draw a tangent TUV to the focal hyperbola, meeting the asymptotes in T and V; then I can shew *geometrically*, as also in other ways,—what indeed, is likely enough to be known,—that the sides of the triangle TAV are, as respects their *lengths*,

$$\overline{\mathrm{AV}} = a + c; \quad \overline{\mathrm{AT}} = a - c; \quad \overline{\mathrm{TV}} = 2b. \tag{3}$$

Now my η and θ are precisely the halves of the sides AV and AT of this triangle; or they are the two co-ordinates of the umbilic U, referred to the two asymptotes, when *directions* as well as lengths are attended to. This explains several of my formulæ, and accounts for the remarkable circumstance that we can pass to a *confocal surface*, by changing η and θ to $t^{-1}\eta$ and $t\theta$ respectively, where t is a scalar.

"Again, we have, identically,

$$\mathrm{V}\frac{\rho\eta - \theta\rho}{\eta - \theta} = \rho_1 + \rho_2; \tag{4}$$

if for conciseness we write

$$\rho_1 = (\eta - \theta)^{-1}\,\mathrm{S}\,.\,(\eta - \theta)\rho; \tag{5}$$

$$\rho_2 = \mathrm{V}\,.\,(\eta - \theta)^{-1}\,\mathrm{V}\,.\,(\eta + \theta)\rho. \tag{6}$$

But ρ_1 is the perpendicular from the centre A of the ellipsoid on the plane of a circular section, through the extremity of any vector or semidiameter ρ; and ρ_2 may be shewn (by a process similar to that which I used to express Mac Cullagh's mode of generation) to be a radius of that circular section, multiplied by the scalar coefficient $\mathrm{S}\,.\,(\eta-\theta)^{-1}(\eta+\theta)$, which is equal to

$$\frac{\theta^2 - \eta^2}{-(\eta - \theta)^2} = \frac{\mathrm{T}\eta^2 - \mathrm{T}\theta^2}{\mathrm{T}(\eta - \theta)^2} = \frac{ac}{b^2}. \tag{7}$$

If, then, from the foot of the perpendicular let fall, as above, on the plane of a circular section, we draw a right line in that plane, which bears to the radius of that section the constant ratio of the rectangle (ac) under the two extreme semiaxes to the square (b^2) of the mean semiaxis of the ellipsoid, the equation (2) expresses that the line so drawn will terminate on a spheric surface, which has its centre at the centre of the ellipsoid, and has its radius $= \dfrac{ac}{b}$; this last being the value of the second member of the equation (2). And, in fact, it is not difficult to prove *geometrically* that this construction conducts to this spheric locus, namely, to the sphere concentric with the ellipsoid, which touches at once the four umbilicar tangent planes."