ON THEOREMS RELATING TO SURFACES, OBTAINED BY THE METHOD OF QUATERNIONS

By

William Rowan Hamilton

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By Sir William R. Hamilton.

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The following letter from Sir William R. Hamilton was read, giving some general expressions of theorems relating to surfaces, obtained by his method of quaternions:

"The equation of curved surface being put under the form

$$f(\rho) = \text{const.}:$$

while its *tangent plane* may be represented by the equation,

$$df(\rho) = 0,$$

or

$$\mathrm{S}\,.\,\nu\, d\rho = 0,$$

if $d\rho$ be the vector drawn to a point of that plane, from the point of contact; the equation of *an osculating surface of the second order* (having complete contact of the second order with the proposed surface at the proposed point) may be thus written:

$$0 = df(\rho) + \tfrac{1}{2} d^2 f(\rho);$$

(by the extension of Taylor's series to quaternions); or thus,

$$0 = 2\mathrm{S}\,.\,\nu\, d\rho + \mathrm{S}\,.\,d\nu\, d\rho,$$

if

$$df(\rho) = 2\mathrm{S}\,.\,\nu\, d\rho.$$

"The *sphere, which osculates in a given direction*, may be represented by the equation

$$0 = 2\mathrm{S}\frac{\nu}{\Delta\rho} + \mathrm{S}\frac{d\nu}{d\rho};$$

where $\Delta\rho$ is a chord of the sphere, drawn from the point of osculation, and

$$\mathrm{S}\frac{d\nu}{d\rho} = \frac{\mathrm{S}\,.\,d\nu\, d\rho}{d\rho^2} = \frac{d^2 f(\rho)}{2\, d\rho^2}$$

is a scalar function of the versor $\mathrm{U}\,d\rho$, which determines the direction of osculation. Hence the important formula:

$$\frac{\nu}{\rho-\sigma} = \mathrm{S}\frac{d\nu}{d\rho};$$

where σ is the vector of the centre of the sphere which osculates in the direction answering to $\mathrm{U}\,d\rho$.

"By combining this with the expression formerly given by me for a normal to the ellipsoid, namely

$$(\kappa^2-\iota^2)^2\nu = (\iota^2+\kappa^2)\rho + \iota\rho\kappa + \kappa\rho\iota,$$

the known value of the curvature of a normal section of that surface may easily be obtained. And for *any* curved surface, the formula will be found to give easily this general theorem, which was perceived by me in 1824; that if, on a normal plane OPP′, which is drawn through a given normal PO, and through any linear element PP′ of the surface, we project the infinitely near normal P′O′, which is erected to the same surface at the end of the element PP′; the projection of the near normal will cross the given normal in the centre O of the same sphere which osculates to the given surface at the given point P, in the direction of the given element PP′.

"I am able to shew that the formula

$$0 = \delta\mathrm{S}\frac{d\nu}{d\rho},$$

which follows from the above, for determining the directions of osculation of the greatest and least osculating spheres, agrees with my formerly published formula,

$$0 = \mathrm{S}\,.\,\nu\,d\nu\,d\rho,$$

for the directions of the lines of curvature.

"And I can deduce Gauss's *general* properties of geodetic lines by showing that if σ_1, σ_2 be the two extreme values of the vector σ, then

$$\frac{-1}{(\rho-\sigma_1)(\rho-\sigma_2)} = \text{measure of curvature of surface} = \frac{1}{R_1R_2} = \frac{d^2\mathrm{T}\,\delta\rho}{\mathrm{T}\,\delta\rho\,.\,d\rho^2};$$

where d answers to motion along a normal section, and δ to the passage from one near (normal) section to another; while S, T, and U, are the characteristics of the operations of taking the scalar, tensor and versor of a quaternion: and the variation δv of the inclination v of a given geodetic line to a variable normal section, obtained by passing from one such section to a near one, without changing the geodetic line, is expressed by the analogous formula,

$$\delta v = -\frac{d\mathrm{T}\,\delta\rho}{\mathrm{T}\,d\rho}.\text{"}$$