ACCOUNT OF THE ICOSIAN CALCULUS

By

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Sir William Rowan Hamilton read a Paper on a new System of Roots of Unity, and of operations therewith connected: to which system of symbols and operations, in consequence of the geometrical character of some of their leading interpretations, he is disposed to give the name of the 'Icosian Calculus.'

This Calculus *agrees* with that of the Quaternions, in three important respects: namely, 1st, that its three chief symbols, ι, κ, λ, are (as above suggested) *roots* of *unity*, as i, j, k are certain *fourth roots* thereof: 2nd, that these new roots *obey* the *associative* law of multiplication; and 3rd, that they are *not* subject to the *commutative* law, or that their *places* as *factors* must *not* in general be *altered* in a *product.* And it *differs* from the Quaternion Calculus, 1st, by involving roots with *different exponents*; and 2nd by *not requiring* so far as yet appears) the *distributive* property of multiplication. In fact, + and −, in these new calculuations, enter *only* as *connecting exponents*, and *not* as connecting *terms*: indeed, *no terms*, or in other words, *no polynomes*, nor even binomes, have hitherto presented themselves, in these late researches of the author. As regards the *exponents* of the new roots, it may be mentioned that in the *principal system*—for the new Calculus involves a *family of systems*—there are adopted the equations,

$$1 = \iota^2 = \kappa^3 = \lambda^5, \quad \lambda = \iota\kappa; \tag{A}$$

so that we deal, in it, with a *new square root, cube root*, and *fifth root*, of *positive unity*; the latter root being the *product* of the two former, when taken in an *order* assigned, but *not* in the opposite order. From these simple assumptions (A), a long train of consistent calculations opens itself out, for every result of which there is found a corresponding geometrical interpretation, in the theory of two of the celebrated solids of antiquity, alluded to with interest by Plato in the Timaeus; namely, the Icosaedron, and the Dodecaedron: whereof the *angles* may *now* be *unequal.* By making $\lambda^4 = 1$, the author obtains other symbolical results, which are interpreted by the Octahedron and the Hexahedron. The Pyramid is, in *this* theory, almost too simple to be interesting: but it is dealt with by the assumption, $\lambda^3 = 1$, the other equations (A) being untouched. As one fundamental result of those equations (A), which may serve as a slight specimen of the rest, it is found that if we make $\iota\kappa^2 = \mu$, we shall have

$$\mu^5 = 1, \quad \mu = \lambda\iota\lambda, \quad \lambda = \mu\iota\mu;$$

so that this *new fifth root* μ has relations of perfect *reciprocity* with the former fifth root λ. But there exist more *general* results, *including* this, and others, on which Sir W. R. H. hopes to be allowed to make a future communication to the Academy: as also on some applications of the principles already stated, or alluded to, which appear to be in some degree interesting.