

**MEMORANDUM RESPECTING A NEW SYSTEM  
OF ROOTS OF UNITY**

**By**

**William Rowan Hamilton**

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*Memorandum respecting a new System of Roots of Unity. By Sir WILLIAM ROWAN HAMILTON, LL.D., M.R.I.A., F.R.A.S., &c., Andrews' Professor of Astronomy in the University of Dublin, and Royal Astronomer of Ireland\*.*

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I have lately been led to the conception of a new system, or rather *family of systems*, of *non-commutative roots of unity*, which are entirely distinct from the *ijk* of the quaternions, though having some general analogy thereto; and which admit, even more easily than the quaternion symbols do, of *geometrical interpretation*. In the system which seems at present to be the most interesting one, among those included in this new family, I assume three symbols,  $\iota$ ,  $\kappa$ ,  $\lambda$ , such that

$$\left. \begin{aligned} \iota^2 = 1, \quad \kappa^3 = 1, \quad \lambda^5 = 1, \\ \lambda = \iota\kappa; \end{aligned} \right\} \quad (\text{A})$$

where  $\iota\kappa$  must be *distinguished* from  $\kappa\iota$ , since otherwise we should have  $\lambda^6 = 1$ ,  $\lambda = 1$ . As a very simple *specimen* of the symbolical conclusions deduced from these fundamental assumptions, I may mention that if we make

$$\mu = \iota\kappa^2 = \lambda\iota\lambda,$$

we shall have also†

$$\mu^5 = 1, \quad \lambda = \mu\mu\mu;$$

so that  $\mu$  is a new fifth root of unity, connected with the former fifth root  $\lambda$  by relations of perfect reciprocity. A long train of such symbolical deductions is found to follow: and every one of the results may be *interpreted*, as having reference to the passage from *face to face* (or from corner to corner) of the *icosahedron* (or of the dodecahedron): on which account, I am at present disposed to give the name of the “Icosian Calculus,” to this new system of symbols, and of rules for their operation. Some additional remarks on this subject may soon

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\* Communicated by the Author.

† In fact, by (A),

$$\begin{aligned} \iota\kappa &= (\iota\kappa)^{-4} = (\kappa^{-1}\iota^{-1})^4 = (\kappa^2\iota)^4, \\ 1 &= \iota \cdot \iota\kappa \cdot \kappa^2 = \iota(\kappa^2\iota)^4\kappa^2 = (\iota\kappa^2)^5; \end{aligned}$$

also

$$\mu\mu\mu = \mu\kappa^2 = \iota\kappa^4 = \iota\kappa = \lambda.$$

be offered to the Philosophical Magazine, under the title, already sanctioned by the Editors,  
of "Extensions of the Quaternions."

Observatory of Trinity College, Dublin,  
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