ON “GAUCHE” CURVES OF THE THIRD DEGREE

By

William Rowan Hamilton

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Sir William Rowan Hamilton

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The following letter, addressed to the President, by Sir W. R. Hamilton, was read:—

Observatory, 27 April 1863.

My dear Mr. President,—I have been wishing for your permission to report, through you, to the Royal Irish Academy, some of the results to which I have lately arrived, while extending the applications of Quaternions, in connexion with my forthcoming Elements.

I. One set of such results relates to those gauche curves of the third degree, which appear to have been first discovered, described, and to some extent applied, by Professor M"obius, in the Barycentric Calculus (1827), and afterwards independently by M. Chasles, in a Note to his Aperçu Historique sur l’Origine et le Développement des Méthodes en Géométrie (1837); and for which our countryman, Dr. Salmon, who has done so much for the Classification of Curves in Space, has proposed the short but expressive name of Twisted Cubics.

II. A particular curve of that class presented itself to me in an investigation more than ten years ago, and some account of it was given in my Lectures, and (I think) to the Academy also, in connexion with the problem of Inscription of Polygons in surfaces of the second order. I gave its vector equation, which was short, but was not sufficiently general, to represent all curves in space of the third degree: nor had I, at the time, any aim at such representation. But I have lately perceived, and printed (in the Elements), the strikingly simple, and yet complete equation,

\[ V\alpha \rho + V\phi \phi \rho = 0, \]

which represents all twisted cubics, if only a point of the curve be taken, for convenience, as the origin: \( \phi \rho \) denoting that linear and vector function of a vector, which has formed the subject of many former studies of mine, and \( \alpha \) being a constant vector, while \( \rho \) is a variable one.

III. It is known that a twisted cubic can in general be so chosen, as to pass through any six points of space. It is therefore natural to inquire, what is the Osculating Twisted Cubic to a given curve of double curvature, or the one which has, at any given place, a six-point contact with the curve. Yet I have not hitherto been able to learn, from any book or
friend, that even the conception of the problem of the determination of such an osculatrix, had occurred to any one before me. But it presented itself naturally to me lately, in the course of writing out a section on the application of quaternions to curves; and I conceive that I have completely resolved it, in three distinct ways, of which two seem to admit of being geometrically described, so as to be understood without diagrams or calculation.

IV. It is known that the cone of chords of a twisted cubic, having its vertex at any one point of that curve, is a cone of the second order, or what Dr. Salmon calls briefly a quadric cone. If, then, a point \( p \) of a given curve in space be made the vertex of a cone of chords of that curve, the quadric cone which has its vertex at \( p \), and has five-side contact with that cone, must contain the osculating cubic sought. I have accordingly determined, by my own methods, the cone which is thus one locus for the cubic: and may mention that I find fifth differentials to enter into its equation, only through the second differential of the second curvature, of the given curve in space. This may perhaps have not been previously perceived, although I am aware that Mr. Cayley and Dr. Salmon, and probably others, have investigated the problem of five-point contact of a plane conic with a plane curve.

V. It is known also that three quadric cylinders can be described, having their generating lines parallel to the three (real or imaginary) asymptotes of a twisted cubic, and wholly containing that gauche curve. My first method, then, consisted in seeking the (necessarily real) direction of one such asymptote, for the purpose of determining a cylinder which, as a second locus, should contain the osculating cubic sought. And I found a cubic cone, as the locus for the generating line (or edge) of such a cylinder, through the given point \( p \) of osculation: and proved that of the six right lines, common to the quadric and the cubic cones, three were absorbed in the tangent to the given curve at \( p \).

VI. In fact, I found that this tangent, say \( pt \), was a nodal side (or ray) of the cubic cone; and that one of the two tangent planes to that cone, along that side, was the osculating plane to the curve, which plane also touched the quadric cone along that common side: while the same side was to be counted a third time, as being a line of intersection, namely, of the quadric cone with the second branch of the cubic cone, the tangent plane to which branch was found to cut the first branch, or the quadric cone, or the osculating plane to the curve, at an angle of which the trigonometric cotangent was equal to half the differential of the radius of second curvature, divided by the differential of the arc of the same given curve.

VII. It might then have been thus expected that a cubic equation could be assigned, of an algebraical form, but involving fifth differentials in its coefficients, which should determine the three planes, tangential to the curve, which are parallel to the three asymptotes of the sought twisted cubic: and then, with the help of what had been previously done, should assign the three quadric cylinders which wholly contain that cubic.

VIII. Accordingly, I succeeded, by quaternions, in forming such a cubic equation, for curves in space generally: and its correctness was tested, by application to the case of the helix, the fact of the six-point contact of my osculating cubic with which well-known curve admitted of a very easy and elementary verification. I had the honour of communicating an
outline of my results, so far, to Dr. Hart, a few weeks ago, with a permission, or rather a request, which was acted on, that he should submit them to the inspection of Dr. Salmon.

IX. Such, then, may be said briefly to have been my first general method of resolving this new problem, of the determination of the twisted cubic which osculates, at a given point, to a given curve of double curvature. Of my second method it may be sufficient here to say, that it was suggested by a recollection of the expressions given by Professor Möbius, and led again to a cubic equation, but this time for the determination of a coefficient, in a development of a comparatively algebraical kind. For the moment I only add, that the second method of solution, above indicated, bore also the test of verification by the helix; and gave me generally fractional expressions for the co-ordinates of the osculating twisted cubic, which admitted, in the case of the helix, of elementary verifications.

X. Of my third general method, it may be sufficient at this stage of my letter to you to say, that it consists in assigning the locus of the vertices of all the quadric cones, which have six-point contact with a given curve in space, at a given point thereof. I find this locus to be a ruled cubic surface, on which the tangent PT to the curve is a singular line, counting as a double line in the intersection of the surface with any plane drawn through it; and such that if the same surface by cut by a plane drawn across it, the plane cubic which is the section has generally a node, at the point where the plane crosses that line: although this node degenerates into a cusp, when the cutting plane passes through the point P itself.

XI. And I find, what perhaps is a new sort of result in these questions, that the intersection of this new cubic surface with the former quadric cone, consists only of the right line PT itself, and of the osculating twisted cubic to the proposed curve in space.

XII. These are only specimens of one set (as above hinted) of recent results obtained through quaternions; but at least they may serve to mark, in some small degree, the respect and affection, to the Academy, and to yourself, with which I remain,

My dear Mr. President,
Faithfully yours,
William Rowan Hamilton

The Very Rev. Charles Graves, D. D., P.R.I.A.,
Dean of the Chapel Royal, &c.