ON DIFFERENCES AND DIFFERENTIALS OF FUNCTIONS OF ZERO

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The first important researches on the differences of powers of zero, appear to be those which Dr. BRINKLEY published in the Philosophical Transactions for the year 1807. The subject was resumed by Mr. HERSCHEL in the Philosophical Transactions for 1816; and in a collection of Examples on the Calculus of Finite Differences, published a few years afterwards at Cambridge. In the latter work, a remarkable theorem is given, for the development of any function of a neperian exponential, by means of differences of powers of zero. In meditating upon this theorem of Mr. HERSCHEL, I have been led to one more general, which is now submitted to the Academy. It contains three arbitrary functions, by making one of which a power and another a neperian exponential, the theorem of Mr. HERSCHEL may be obtained.

Mr. HERSCHEL'S Theorem is the following:

$$f(e^t) = f(1) + tf(1+\Delta)o^1 + \frac{t^2}{1 \,.\, 2} f(1+\Delta)o^2 + \&\text{c.} \tag{A}$$

$f(1+\Delta)$ denoting any function which admits of being developed according to positive integer powers of Δ, and every product of the form $\Delta^m o^n$ being interpreted, as in Dr. BRINKLEY'S notation, as a difference of a power of zero.

The theorem which I offer as a more general one may be thus written:

$$\phi(1+\Delta) f\psi(o) = f(1+\Delta')\phi(1+\Delta)(\psi(o))^{o'}; \tag{B}$$

or thus

$$F(D) f\psi(o) = f(1+\Delta') F(D)(\psi(o))^{o'}. \tag{C}$$

In these equations, f, ϕ, F, ψ, are arbitrary functions, such however that $f(1+\Delta')$, $\phi(1+\Delta)$, $F(D)$, can be developed according to positive integer powers of Δ' Δ D; and after this development Δ' Δ are considered as marks of differencing, referred to the variables o' o, which vanish after the operations, and D as a mark of derivation by differentials, referred to the variable o. And if in the form (C) we particularise the functions F, ψ, by making F a power, and ψ a neperian exponential, we deduce the following corollary:

$$D^x f(e^o) = f(1+\Delta') D^x e^{o'} = f(1+\Delta') o'^x;$$

that is, the coefficient of $\dfrac{t^x}{1 \,.\, 2 \ldots x}$ in the development of $f(e^t)$ may be represented by $f(1+\Delta)o^x$; which is the theorem (A) of Mr. HERSCHEL.

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ADDITION.

The two forms (B) (C) may be included in the following:

$$\nabla' f \psi(o') = f(1+\Delta)\nabla'(\psi(o'))^o. \tag{D}$$

To explain and prove this equation, I observe that in MACLAURIN'S series,

$$f(x) = f(o) + \frac{Df(o)}{1}x + \frac{D^2 f(o)}{1\,.\,2}x^2 + \cdots + \frac{D^n f(o)}{1\,.\,2\,\ldots\,n}x^n + \cdots$$

we may put $x = (1+\Delta)x^o$ and therefore may put the series itself under the form

$$f(x) = f(o) + \frac{Df(o)}{1}(1+\Delta)x^o + \frac{D^2 f(o)}{1\,.\,2}(1+\Delta)^2 x^o + \&\text{c.}$$

or more concisely thus

$$f(x) = f(1+\Delta)x^o : \tag{E}$$

which latter expression is true even when MACLAURIN'S series fails, and which gives, by considering x as a function ψ of a new variable o' and performing any operation ∇' with reference to the latter variable,

$$\nabla' f \psi(o') = \nabla' f(1+\Delta)(\psi(o'))^o. \tag{F}$$

If now the operation ∇' consist in any combination of differencings and differentiatings, as in the equations (B) and (C), and generally if we may transpose the symbols of operation ∇' and $f(1+\Delta)$, which happens for an infinite variety of forms of ∇', we obtain the theorem (D). It is evident that this theorem may be extended to functions of several variables.

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