ON FLUCTUATING FUNCTIONS
(ABSTRACT)

By

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Sir William Rowan Hamilton

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The President gave an account of some investigations respecting *Fluctuating Functions*, from which the following are extracts:—

"Let P_x denote any real function [of] x, continuous or discontinuous, but such that its first and second integrals,

$$\int_0^x dx\, \mathrm{P}_x, \quad \text{and} \quad \left(\int_0^x dx\right)^2 \mathrm{P}_x,$$

are always comprised between given finite limits. Let also the equation

$$\left(\int_0^x dx\right)^2 \mathrm{P}_x = \mu,$$

in which μ is some given constant, have infinitely many real roots, both positive and negative, which are not themselves comprised between any finite limits, but are such that the interval between one and the next greater root is never greater than some given finite interval. Then

$$\lim_{t=\infty} \int_a^b dx \int_0^{tx} dy\, \mathrm{P}_y \mathrm{F}_x = 0, \tag{A}$$

if a and b are any finite values of x, between which the function F_x is finite.

"Again, the same things being supposed, let the arbitrary function F_x vary gradually between the same values of x, and let P_x be finite and vary gradually when x is infinitely small; then

$$\mathrm{F}_y = \varpi^{-1} \int_0^\infty dt \int_a^b dx\, \mathrm{P}_{tx-ty} \mathrm{F}_x, \quad \left(y \begin{matrix} > a \\ < b \end{matrix}\right), \tag{B}$$

in which

$$\varpi = \int_{-\infty}^\infty dx \int_0^1 \mathrm{P}_{tx}.$$

"For the case $y = a$, we must change ϖ, in (B), to

$$\varpi' = \int_0^\infty dx \int_0^1 dt\, \mathrm{P}_{tx};$$

and for the case $y = b$, we must change it to

$$\varpi'' = \int_{-\infty}^{0} dx \int_{0}^{1} dt \, \mathrm{P}_{tx}.$$

“For values of $y > b$, or $< a$, the second member of the formula (B) vanishes.

“If F_x, although finite, were to receive any sudden change for some particular value of y between a and b, so as to pass suddenly from the value F'' to the value F', we should then have, for this value of y,

$$\int_{0}^{\infty} dt \int_{a}^{b} dx \, \mathrm{P}_{tx-ty} \mathrm{F}_x = \varpi' \mathrm{F}' + \varpi'' \mathrm{F}''.$$

By changing P_x to $\cos x$, we obtain from (B) the celebrated theorem of Fourier. Indeed, that great mathematician appears to have possessed a clear conception of the *principles* of fluctuating functions, although he is not known to have deduced from them consequences so general as the above.

“Again another celebrated theorem is comprised in the following:—

$$\mathrm{F}_y = \varpi^{-1} \mathrm{P}_0 \left(\int_{a}^{b} dx \, \mathrm{F}_x + \sum\nolimits_{(n)1}^{\infty} \int_{a}^{b} dx \, \mathrm{Q}_{x-y,n} \mathrm{F}_x \right), \tag{C}$$

in which, the function Q is defined by the conditions

$$\mathrm{Q}_{x,n} \int_{0}^{x} dx \, \mathrm{P}_x = \int_{2nx-x}^{2nx+x} dx \, \mathrm{P}_x;$$

y is $> a$, $< b$; and no real root of the equation

$$\int_{0}^{\infty} dx \, \mathrm{P}_x = 0,$$

except the root 0 is included between the negative number $a - y$ and the positive number $b - y$, nor are those numbers themselves supposed to be roots of that equation. When these conditions are not satisfied, the theorem (C) takes other forms, which, with other analogous results, may be deduced from the same principles.”