RESEARCHES RESPECTING VIBRATION, CONNECTED WITH THE THEORY OF LIGHT

By

William Rowan Hamilton

(Proceedings of the Royal Irish Academy, 1 (1841), pp. 341–349.)

Edited by David R. Wilkins

2000

Researches respecting Vibration, connected with the Theory of Light.

By Sir William R. Hamilton.

Communicated June 24, 1839.

[*Proceedings of the Royal Irish Academy*, vol. 1 (1841), pp. 341–349.]

The President concluded his account of his First Series of Researches respecting Vibration, connected with the Theory of Light. The following is an outline of one of the investigations which are contained in the Series referred to.

It is proposed to integrate the system of equations in mixed differences,

$$\mathrm{D}_t^2\,\delta x_{g,h} = \Sigma_{\Delta g}\delta(\mathrm{R}\,.\,\Delta_g x_{g,h}); \tag{1}$$

in which h is any integer number from 1 to n inclusive; $x_{g,h}$ is independent of t, but $\delta x_{g,h}$ is a function of t and of $x_{g,1}, \ldots\ x_{g,n}$, the form of which function it is the object of the problem to discover;

$$\mathrm{R} = m_{g+\Delta g}\phi\left(\tfrac{1}{2}\Sigma_{(h)}{}_1^n(\Delta_g x_{g,h})^2\right), \tag{2}$$

ϕ being any real function of the semi-sum which follows it, and m being any other real function of the index $g+\Delta g$; while g and $g+\Delta g$ represent any integer numbers from negative to positive infinity. The equations to be integrated may also be thus written:

$$\xi''_{g,h,t} = \Sigma_{\Delta g}\left(\mathrm{R}\,\Delta_g\xi_{g,h,t} + \mathrm{R}'\,\Delta_g x_{g,h}\Sigma_{(h)}{}_1^n\Delta_g x_{g,h}\Delta_g\xi_{g,h,t}\right), \tag{1$'$}$$

in which

$$\mathrm{R}' = m_{g+\Delta g}\,\phi'\left(\tfrac{1}{2}\Sigma_{(h)}{}_1^n(\Delta_g x_{g,h})^2\right), \tag{2$'$}$$

the functions to be found by integration are now those of the form $\xi_{g,h,t}$, considered as depending on t and on $x_{g,1}, \ldots\ x_{g,n}$; their initial values, and initial rates of increase (relatively to t), namely $\xi_{g,h,0}$ and $\xi'_{g,h,0}$, are regarded as arbitrary but given and real functions of $x_{g,1}, \ldots\ x_{g,n}$; it is also supposed, in order to simplify the question, that all the sums of the forms

$$\Sigma_{\Delta g}\mathrm{R}(\Delta_g x_{g,1})^{\alpha_1}\ \ldots\ (\Delta_g x_{g,n})^{\alpha_n}, \quad \Sigma_{\Delta g}\mathrm{R}'(\Delta_g x_{g,1})^{\alpha_1}\ \ldots\ (\Delta_g x_{g,n})^{\alpha_n}, \tag{3}$$

are independent of g, and are $= 0$ when any one of the exponents $\alpha_1, \ldots\ \alpha_n$ is an odd number. These equations are analogous to, and include, those which M. Cauchy has considered on his memoir on the Dispersion of Light, and may be integrated by a similar analysis.

A particular integral system may in the first case be found by assuming

$$\xi_{g,h,t} = \mathrm{X}_r\mathrm{A}_{h,r}\cos(\epsilon_r + s_r t - \Sigma_{(i)}{}_1^n u_i x_{g,i}); \tag{4}$$

$$\Sigma_{(h)}{}_1^n\mathrm{A}_{h,r}^2 = 1; \tag{5}$$

$$s_r^2 \mathrm{A}_{h,r} = \Sigma_{(i)}\,{}_1^n \mathrm{H}_{h,i}\mathrm{A}_{i,r}; \tag{6}$$

$$\mathrm{H}_{h,h} = \Sigma_{\Delta g}\left(\mathrm{R} + \mathrm{R}'(\Delta_g x_{g,h})^2\right) \text{vers}\left(\Sigma_{(i)}\,{}_1^n u_i\, \Delta_g x_{g,i}\right), \tag{7}$$

$$\mathrm{H}_{h,i} = \Sigma_{\Delta g}\mathrm{R}'\, \Delta_g x_{x,h}\, \Delta_g x_{g,i}\, \text{vers}\left(\Sigma_{(i)}\,{}_1^n u_i\, \Delta_g x_{g,i}\right); \tag{7$'$}$$

the index r being any integer from 1 to n, and being introduced in order to distinguish among themselves the n different (and in general real) systems of values of s^2, and of the $n-1$ ratios of $\mathrm{A}_1, \dots \mathrm{A}_h, \dots \mathrm{A}_n$, which are obtained by resolving the system of the n equations of the form

$$s^2 \mathrm{A}_h = \Sigma_{(i)}\,{}_1^n \mathrm{H}_{h,i}\mathrm{A}_i, \tag{6$'$}$$

in which, by (7)$'$,

$$\mathrm{H}_{i,h} = \mathrm{H}_{h,i}. \tag{7$''$}$$

It is important to observe, that by the form of these equations (6)$'$, (which occur in may researches,) we have the relation

$$\Sigma_{(h)}\,{}_1^n \mathrm{A}_{h,q}\mathrm{A}_{h,r} = 0, \tag{5$'$}$$

if q be different from r, and that, by (5) and (5)$'$, we have also the relations

$$\Sigma_{(r)}\,{}_1^n \mathrm{A}_{h,r}^2 = 1, \tag{8}$$

$$\Sigma_{(r)}\,{}_1^n \mathrm{A}_{h,r}\mathrm{A}_{i,r} = 0. \tag{8$'$}$$

In the particular integral (4), we may consider $u_1, \dots u_n$ as arbitrary parameters, of which X_r and ϵ_r are real and arbitrary, while s_r^2 and $\mathrm{A}_{h,r}$ are real and determined functions; and hence, by summations relatively to the index r, and integrations relatively to the parameters u_i, employing also the relations (5) (5)$'$ (8) (8)$'$, and Fourier's theorem extended to several variables, we deduced this general integral, applying to all arbitrary real values of the initial data:

$$\xi_{g,h,t} = \left(\Pi_{(i)}\,{}_1^n \int_{-\infty}^{\infty} du_i\right)(\mathrm{E}_{h,t}\cos + \mathrm{F}_{h,t}\sin)\,\Sigma_{(i)}\,{}_1^n u_x x_{g,i}; \tag{9}$$

in which

$$\Pi_{(i)}\,{}_1^n \int_{-\infty}^{\infty} du_i = \int_{-\infty}^{\infty} du_1 \int_{-\infty}^{\infty} du_2 \dots \int_{-\infty}^{\infty} du_n; \tag{10}$$

$$\left.\begin{aligned} \mathrm{E}_{h,t} &= \Sigma_{(r)}\,{}_1^n \mathrm{A}_{h,r}\left(\mathrm{Y}_r \cos ts_r + \mathrm{Y}'_r s_r^{-1} \sin ts_r\right), \\ \mathrm{F}_{h,t} &= \Sigma_{(r)}\,{}_1^n \mathrm{A}_{h,r}\left(\mathrm{Z}_r \cos ts_r + \mathrm{Z}'_r s_r^{-1} \sin ts_r\right); \end{aligned}\right\} \tag{11}$$

$$\left.\begin{aligned} \mathrm{Y}_r &= \Sigma_{(h)}\,{}_1^n \mathrm{A}_{h,r}\mathrm{E}_{h,0}, \quad & \mathrm{Y}'_r &= \Sigma_{(h)}\,{}_1^n \mathrm{A}_{h,r}\mathrm{E}'_{h,0}, \\ \mathrm{Z}_r &= \Sigma_{(h)}\,{}_1^n \mathrm{A}_{h,r}\mathrm{F}_{h,0}, \quad & \mathrm{Z}'_r &= \Sigma_{(h)}\,{}_1^n \mathrm{A}_{h,r}\mathrm{F}'_{h,0}; \end{aligned}\right\} \tag{12}$$

$$\left.\begin{aligned}
\mathrm{E}_{h,0} &= \left(\frac{1}{2\pi}\right)^n \left(\Pi_{(i)}\,{}_1^n \int_{-\infty}^{\infty} dx_{g,i}\right) \xi_{g,h,0} \cos\left(\Sigma_{(i)}\,{}_1^n u_i x_{g,i}\right), \\
\mathrm{E}'_{h,0} &= \left(\frac{1}{2\pi}\right)^n \left(\Pi_{(i)}\,{}_1^n \int_{-\infty}^{\infty} dx_{g,i}\right) \xi'_{g,h,0} \cos\left(\Sigma_{(i)}\,{}_1^n u_i x_{g,i}\right), \\
\mathrm{F}_{h,0} &= \left(\frac{1}{2\pi}\right)^n \left(\Pi_{(i)}\,{}_1^n \int_{-\infty}^{\infty} dx_{g,i}\right) \xi_{g,h,0} \sin\left(\Sigma_{(i)}\,{}_1^n u_i x_{g,i}\right), \\
\mathrm{F}'_{h,0} &= \left(\frac{1}{2\pi}\right)^n \left(\Pi_{(i)}\,{}_1^n \int_{-\infty}^{\infty} dx_{g,i}\right) \xi'_{g,h,0} \sin\left(\Sigma_{(i)}\,{}_1^n u_i x_{g,i}\right).
\end{aligned}\right\} \tag{13}$$

This general solution involves multiple integrals, of the order $2n$; but many particular suppositions, respecting the initial data, conduct to simpler expressions, among which the following appear worthy of remark.

Suppose that having assumed some particular set $u^{\backprime}_1, \ldots u^{\backprime}_n$ of values of the n arbitrary quantities $u_1, \ldots u_n$, we deduce a corresponding set of coefficients $\mathrm{H}^{\backprime}_{h,h}$, $\mathrm{H}^{\backprime}_{h,i}$, by the formulæ (7) and (7)$'$, and represent by $s^{\backprime 2}_1$ and by $\mathrm{A}^{\backprime}_{1,1}, \ldots \mathrm{A}^{\backprime}_{h,1}, \ldots \mathrm{A}^{\backprime}_{n,1}$ some one corresponding system of quantities which satisfy the equations

$$\Sigma_{(h)}\,{}_1^n \mathrm{A}^{\backprime 2}_{h,1} = 1, \tag{5$^{\backprime}$}$$

$$s^{\backprime 2}_1 \mathrm{A}^{\backprime}_{h,1} = \Sigma_{(i)}\,{}_1^n \mathrm{H}^{\backprime}_{h,i} \mathrm{A}^{\backprime}_{i,1}; \tag{6$^{\backprime}$}$$

we shall then have, as a particular integral system, that which is thus denoted:

$$\xi_{g,h,t} = \mathrm{X}^{\backprime}_1 \mathrm{A}^{\backprime}_{h,1} \cos(\epsilon^{\backprime}_1 + s^{\backprime}_1 t - \Sigma_{(i)}\,{}_1^n u^{\backprime}_i x_{g,i}); \tag{4$^{\backprime}$}$$

$\mathrm{X}^{\backprime}_1$ and $\epsilon^{\backprime}_1$ denoting here any arbitary real quantities. If therefore we suppose that the initial data $\xi_{g,h,0}$ and $\xi'_{g,h,0}$ are all such as to agree with this particular solution, that is, if we have for all values of g and h,

$$\xi_{g,h,0} = \mathrm{X}^{\backprime}_1 \mathrm{A}^{\backprime}_{h,1} \cos(\epsilon^{\backprime}_1 - \Sigma_{(i)}\,{}_1^n u^{\backprime}_i x_{g,i}), \tag{14}$$

$$\xi'_{g,h,0} = -s^{\backprime}_1 \mathrm{X}^{\backprime}_1 \mathrm{A}^{\backprime}_{h,1} \sin(\epsilon^{\backprime}_1 - \Sigma_{(i)}\,{}_1^n u^{\backprime}_i x_{g,i}), \tag{14$'$}$$

we see, *à priori*, that the multiple integrations ought to admit of being all effected in finite terms, so as to reduce the general expression (9) to the particular form (4)$^{\backprime}$; an expectation which the calculation, accordindingly, *à posteriori*, proves to be correct. An analogous but less simple reduction takes place, when we suppose that the initial equations (14) and (14)$'$ hold good, after their second members have been multiplied by a discontinuous factor such as

$$\tfrac{1}{2}\left(1 - \frac{2}{\pi}\int_0^{\infty} \frac{\sin\left(k\Sigma_{(i)}\,{}_1^n u^{\backprime}_i x_{g,i}\right)}{k}\, dk\right), \tag{15}$$

which is $= 1$, or $= \frac{1}{2}$, or $= 0$, according as the sum $\Sigma_{(i)}\,{}_1^n u^{\backprime}_i x_{g,i}$ is < 0, $= 0$, or > 0. It is found that, in this case, the $2n$ successive integrations (required for the general solution) can in part be completely effected, and in the remaining part be reduced to the calculation of a simple definite integral; in such a manner that the expression (9) now reduces itself rigorously to the following:

$$\xi_{g,h,t} = \tfrac{1}{2}\mathrm{X}^\backprime_1\mathrm{A}^\backprime_{h,1}\cos(\epsilon^\backprime_1 + ts^\backprime_1 - \Sigma_{(i)}\,{}^n_1 u^\backprime_i x_{g,i}); + \frac{1}{\pi}\mathrm{X}^\backprime_1 \int_0^\infty \frac{dk}{k^2 - k^{\backprime 2}}(\mathrm{L}_t\cos\epsilon^\backprime_1 + \mathrm{M}_t\sin\epsilon^\backprime_1); \quad (16)$$

in which

$$\left.\begin{aligned} \mathrm{L}_t &= \mathrm{P}_t k^\backprime \cos kx - \mathrm{Q}_t k\sin kx, \\ \mathrm{M}_t &= \mathrm{P}_t k\sin kx + \mathrm{Q}_t k^\backprime \cos kx, \end{aligned}\right\} \quad (17)$$

$$\left.\begin{aligned} \mathrm{P}_t &= s^\backprime_1 \Sigma_{(r)}\,{}^n_1(\mathrm{A}_{h,r}s_r^{-1}\sin ts_r\,.\,\Sigma_{(h)}\,{}^n_1\mathrm{A}_{h,r}\mathrm{A}^\backprime_{h,1}), \\ \mathrm{Q}_t &= \Sigma_{(r)}\,{}^n_1(\mathrm{A}_{h,r}\cos ts_r\,.\,\Sigma_{(h)}\,{}^n_1\mathrm{A}_{h,r}\mathrm{A}^\backprime_{h,1}), \end{aligned}\right\} \quad (18)$$

$$x = \Sigma_{(i)}\,{}^n_1 a^\backprime_i x_{g,i}, \quad (19)$$

$$ka^\backprime_i = u_i, \quad k^\backprime a^\backprime_i = u^\backprime_i, \quad k^{\backprime 2} = \Sigma_{(i)}\,{}^n_1 u^{\backprime 2}_i, \quad (20)$$

and s_r, $\mathrm{A}_{h,r}$ are the same functions as before of $u_1, \dots u_n$.

A remarkable conclusion may now be drawn from these expressions, by supposing that all the quantities of the form s_r^2 are not only real but positive, so that the functions $\cos ts_r$ and $\sin ts_r$ are periodic. For in this case the functions $\cos(ts_r \pm kx)$ and $\sin(ts_r \pm kx)$, will vary rapidly, and pass often through all their fluctuations of value, between the limits 1 and -1, while k and the other functions of that variable remain almost unchanged, provided that $t\dfrac{ds_r}{dk} \pm x$ is large, and that the denominator $k^2 - k^{\backprime 2}$ is not extremely small. We may therefore in general confine ourselves to the consideration of small values of this denominator; and consequently may put it under the form $2k^\backprime(k - k^\backprime)$, making $k = k^\backprime$ in the numerator, except under the periodical signs, and integrating relatively to k between any two limits which include $k^\backprime$, for example between $-\infty$ and $+\infty$. And because

$$\Sigma_{(h)}\,{}^n_1\mathrm{A}^\backprime_{h,r}\mathrm{A}^\backprime_{h,1} = 1, \text{ or } = 0,$$

according as $r = 1$ or > 1, we may make

$$\mathrm{P}_t = \mathrm{A}^\backprime_{h,1}\sin ts_1, \quad \mathrm{Q}_t = \mathrm{A}^\backprime_{h,1}\cos ts_1,$$

$$\mathrm{L}_t = k^\backprime\mathrm{A}^\backprime_{h,1}\sin(ts_1 - kx), \quad \mathrm{M}_t = k^\backprime\mathrm{A}^\backprime_{h,1}\cos(ts_1 - kx)$$

and

$$\xi_{g,h,t} = \tfrac{1}{2}\mathrm{X}^\backprime_1\mathrm{A}^\backprime_{h,1}\left\{\cos(\epsilon^\backprime_1 + ts^\backprime_1 - k^\backprime x) + \int_{-\infty}^{\infty} dk\frac{\sin(\epsilon^\backprime_1 + ts_1 - kx)}{\pi(k - k^\backprime)}\right\}, \quad (21)$$

that is, nearly, if x be considerably different from $t\dfrac{ds^\backprime_1}{dk^\backprime}$,

$$\xi_{g,h,t} = \tfrac{1}{2}\mathrm{X}^\backprime_1\mathrm{A}^\backprime_{h,1}\cos(\epsilon^\backprime_1 + ts^\backprime_1 - k^\backprime x)\left\{1 + \int_{-\infty}^{\infty}\frac{dk}{\pi k}\left(\left(t\frac{ds^\backprime_1}{dk^\backprime} - x\right)k\right)\right\}. \quad (21)'$$

We have therefore the approximate expressions:

$$\xi_{g,h,t} = \mathrm{X}^{\backprime}_1 \mathrm{A}^{\backprime}_{h,1} \cos(\epsilon^{\backprime}_1 + ts^{\backprime}_1 - k^{\backprime} x), \quad \text{if } x < t\frac{ds^{\backprime}_1}{dk^{\backprime}}; \tag{22}$$

and

$$\xi_{g,h,t} = 0, \quad \text{if } x > t\frac{ds^{\backprime}_1}{dk^{\backprime}}; \tag{22$'$}$$

we have also nearly, in general,

$$\xi_{g,h,t} = \tfrac{1}{2}\mathrm{X}^{\backprime}_1 \mathrm{A}^{\backprime}_{h,1} \cos(\epsilon^{\backprime}_1 + ts^{\backprime}_1 - k^{\backprime} x), \quad \text{if } x = t\frac{ds^{\backprime}_1}{dk^{\backprime}}; \tag{22$''$}$$

but the discussion of the case when x is nearly $= t\dfrac{ds^{\backprime}_1}{dk^{\backprime}}$ is too long to be cited here. The formula (22) for $\xi_{g,h,t}$ coincides with the particular integral (4)$^{\backprime}$; and the condition which it involves with respect to x, expresses the law according to which this particular integral comes to be (nearly) true for greater and greater positive values of x and t (if $\dfrac{ds^{\backprime}_1}{dk^{\backprime}} > 0$,) after having been true only for negative valus of x when t was $= 0$.

In the particular case $n = 3$, the foregoing formulæ have an immediate dynamical application, and correspond to the propagation of vibratory motion through a system of mutually attracting or repelling particles; and they conduct to this remarkable result, that the velocity with which such vibration spreads into those portions of the vibratory medium which were previously undisturbed, is in general different from the velocity of a passage of a given phase from one particle to another within that portion of the medium which is already fully agitated; since we have

$$\textit{velocity of transmission of phase} = \frac{s}{k}, \tag{A}$$

but

$$\textit{velocity of propagation of vibratory motion} = \frac{ds}{dk}, \tag{B}$$

if the rectangular components of the vibrations themselves be represented by the formulæ

$$\mathrm{XA}_1 \cos(\epsilon + st - kx), \quad \mathrm{XA}_2 \cos(\epsilon + st - kx), \quad \mathrm{XA}_3 \cos(\epsilon + st - kx), \tag{C}$$

t being the time, and x the perpendicular distance of the vibrating point from some determined plane.

This result, which is believed to be new, includes as a particular case that which was stated in a former communication to the Academy, on the 11th of February last, (Proceedings, No. 15, page 269,) respecting the propagation of transversal vibration along a row of equal and equidistant particles, of which each attracts the two that are immediately before and behind it; in which particular question s was $= 2a \sin \dfrac{k}{2}$, and the velocity of propagation of vibration was $= a \cos \frac{k}{2}$. Applied to the theory of light, it appears to show that if the phase of vibration in an ordinary dispersive medium be represented for some one colour by

$$\epsilon + \frac{2\pi}{\lambda}\left(\frac{t}{\mu} - x\right), \tag{C$'$}$$

so that λ is the length of an undulation for that colour and for that medium, and if it be permitted to represent dispersion by developing the velocity $\frac{1}{\mu}$ of the transmission of phase in a series of the form

$$\frac{1}{\mu} = \mathrm{M}_0 - \mathrm{M}_1\left(\frac{2\pi}{\lambda}\right)^2 + \mathrm{M}_2\left(\frac{2\pi}{\lambda}\right)^4 - \&\text{c.}, \tag{A$'$}$$

then the *velocity wherewith light of this colour conquers darkness*, in this dispersive medium, by the *spreading of vibration into parts which were not vibrating before, is somewhat less than* $\frac{1}{\mu}$, being represented by this other series

$$\mathrm{M}_0 - 3\mathrm{M}_1\left(\frac{2\pi}{\lambda}\right)^2 + 5\mathrm{M}_2\left(\frac{2\pi}{\lambda}\right)^4 - \&\text{c.} \tag{B$'$}$$

For other details of this inquiry it is necessary to refer to the memoir itself, which will be pubished in the Transactions of the Academy, and will be found to contain many other investigations respecting vibratory systems, with applications to the theory of light.